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5` [ YVfUjW7\ UfUWYf]nUh]cb`cZ`Gca Y`7`UggjWj`7ca V]bUhcf]U`Df`cV`Ya`g  
5i`h\cf`fj`9`H`CfXa`Ub  
Gci`fW`H\`Y`5a`Yf]Wb`A`Uh\`Ya`Uh]Wj`A`cbh\`mž`J`c`+`ž`B`c`-`fB`cj`"ž`%`+%ž`dd`-`\*`%`-`+`\$  
Di`V`g\`YX`Vm`A`Uh\`Ya`Uh]Wj`5ggcV]U]cb`cZ`5a`Yf]Wj  
GhU`Y`I`F`@`<http://www.jstor.org/stable/2317800>  
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you nor I nor anybody else knows what makes a mathematician tick. It is not a question of cleverness. As I have already said, I know many mathematicians who are far abler and cleverer than I am, but they have not been so lucky. An illustration may be given by considering two miners. One may be an expert geologist, but he does not find the golden nuggets that the ignorant miner does.

In some ways, a mathematician is not responsible for his activities. One sometimes feels there is an inner self occasionally communicating with the outer man. This view is supported by the statements made by H. Poincaré and J. Hadamard about their researches. I remember once walking down St. Andrews Street some three weeks after writing a paper. Though I had never given the matter any thought since then, it suddenly occurred to me that a point in my proof needed looking into.

I am very grateful to my inner self for his valuable help in the solution of some important and difficult problems that I could not have done otherwise.

I commenced this talk by saying a toast had been drunk to me by the Master and Fellows of St. John's College. I might conclude by reciting one sent to me by Professor L. Moser. Of him, it was said that he was writing a book and taking so long about it that his publishers became very much worried and went to see him. He said he was very sorry about the delay, but he was afraid that the book might have to be a posthumous one. Well, he was told, please hurry up with it.

Moser's toast was as follows:

*Here's a toast to L. J. Mordell,  
young in spirit, most active as well,  
He'll never grow weary,  
of his love, number theory,  
The results he obtains are just swell.*

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## ALGEBRAIC CHARACTERIZATION OF SOME CLASSICAL COMBINATORIAL PROBLEMS

E. T. ORDMAN, University of Kentucky

**1. Introduction.** The numbers  $a_n$ ,

$$(1) \quad a_n = \frac{1}{n} \binom{2n-2}{n-1},$$

known as the **Catalan numbers**, occur in a wide variety of combinatorial problems. For example,  $a_n$  is the number of elements in the sets  $A_n$ ,  $E_n$ ,  $S_n$ , where:

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Professor Ordman received his A.B. from Kenyon College, and his Princeton Ph.D. in 1969, having worked under John Stallings at the Univ. of Calif., Berkeley during 1967-69, supported by a Danforth Fellowship. Since 1969 he has been Assistant Professor at the Univ. of Kentucky. His main research is in group theory problems motivated by topology. *Editor.*

$A_n$  is the collection of noncommutative nonassociative binary products of a single generator taken  $n$  times. For instance,

$$A_4 = \{a(a(aa)), a((aa)a), (aa)(aa), (a(aa))a, ((aa)a)a\},$$

so  $a_4 = 5$ .

$E_n$  is the collection of ways in which a fixed regular  $(n+1)$ -gon in the plane may be divided into triangles by  $n-2$  diagonals which do not intersect in its interior.

$S_n$  is the collection of sequences of  $2n-2$  terms  $(x_1, x_2, \dots, x_{2n-2})$ , where each  $x_i = \pm 1$ , subject to the conditions  $x_1 + x_2 + \dots + x_{2n-2} = 0$  and  $x_1 + x_2 + \dots + x_k \geq 0$  for  $1 \leq k \leq 2n-2$ .

We call a set  $Q = Q_1 \cup Q_2 \cup Q_3 \dots$  of Catalan type if formula (1) gives the number of elements of  $Q_n$ . Proofs that various sets (including the above) are of Catalan type appear for instance in [1], [4], [5, Problem 7], [6], and [10, Problems 54, 83, 84]. Our object is to study the algebraic structure underlying sets of Catalan type, in order to help recognize such sets and answer some related questions. We shall not reproduce a proof that formula (1) gives the number of elements of  $A_n$ , but we shall prove that  $A_n, E_n$ , and  $S_n$  may be placed in one-one correspondence.

1.2. Published proofs that a given set  $Q$  is of Catalan type sometimes set up a one-one map between  $Q_n$  and  $A_n$  or some other set known to have the desired number of elements; more often they establish that  $Q_1$  has one element and establish the recurrence (2)

$$(2) \quad \sum q_k q_{n-k} = q_n \quad (\text{sum for } 1 \leq k \leq n-1),$$

where  $Q_n$  has  $q_n$  elements. Since  $a_1 = 1$  and the  $a_i$  satisfy this recurrence,  $q_n = a_n$  by induction. Recurrence (2) is most often established by providing a way of "factoring" an element of  $Q_n$  uniquely as an element of  $Q_k$  and an element of  $Q_{n-k}$ , for some  $k$ . Thus the set  $Q$  is provided with a multiplication. For instance, multiplication in  $A = A_1 \cup A_2 \cup \dots$  is given by  $(b, c) \rightarrow (b)(c)$  for every  $b$  and  $c$  in  $A$  (the parentheses around  $b$  or  $c$  being omitted if it is in  $A_1$ ). In the future, we shall write the product of  $b$  and  $c$  simply as  $bc$ , when no confusion will result.

1.3. The sets  $A, E = E_1 \cup E_2 \cup \dots$ , and  $S = S_1 \cup S_2 \cup \dots$  have various actions on them. We give some examples to be discussed later:

The operation "Mirror image" is a map  $M: A \rightarrow A$ . For instance  $M$  interchanges  $(a(aa))a$  and  $a((aa)a)$ ; but note  $M((aa)aa) = (aa)(aa)$ . Clearly  $M^2 = I$ , the identity map on  $A$ .

The operation "reflection in a vertical line" is a map  $\hat{M}: E \rightarrow E$ . Like  $M$ , it is of order 2. The operation "rotation counterclockwise through  $2\pi/(n+1)$  radians" is a map  $\rho_n: E_n \rightarrow E_n$ , with the property that  $\rho_n^{n+1}$  is the identity map of  $E_n$ .

$S$  has a conspicuous map  $\lambda: S \rightarrow S$  of order 2, given by

$$\lambda(x_1, x_2, \dots, x_{2n-2}) = (-x_{2n-2}, -x_{2n-3}, \dots, -x_1).$$

For instance,  $\lambda(+1, +1, -1, -1, +1, -1) = (+1, -1, +1, +1, -1, -1)$ .

1.4. *Problems.* Two of the maps of 1.3, as well as other “reasonable” maps we shall not study, fail to have various desirable properties. For instance, for none of the published “multiplications”  $f: E_n \times E_m \rightarrow E_{n+m}$  does a relation such as

$$(3) \quad \rho_{n+m} f(b, c) = f(\rho_n b, \rho_m c),$$

for all  $b$  in  $E_n$  and  $c$  in  $E_m$  hold. Might  $f$  be defined to make (3) hold?

It is easy to see that no one-one onto map between  $A_3$  and  $S_3$  can carry  $M$  to  $\lambda$ ; for  $M$  interchanges  $(aa)a$  with  $a(aa)$ , while  $\lambda$  preserves both  $(+1, +1, -1, -1)$  and  $(+1, -1, +1, -1)$ . Is there a “natural” map  $M_S: S \rightarrow S$  corresponding to  $M$ ? How does it relate to  $\lambda$ ?

1.5. Section 2 is a discussion of the algebraic structure  $A$ , characterizing the structure itself, its automorphisms, and isomorphisms between it and other structures. Section 3 applies these methods to  $E$ , solving the problem about  $\rho_n$  stated above (Theorem 3.3). Section 4 contains a discussion of  $S$ , in particular the relation between  $M$  and  $\lambda$ .

1.6. *History.* The sets  $A_n, E_n, S_n$ , and many others like them, have long been well known. Dörrrie [5, Problem 7] traces  $E_n$  back to a 1751 problem of Euler and  $A_n$  to an 1838 paper of Catalan. By 1859 Cayley [4] observed the connection between  $A_n$  and the problem of enumerating certain graphs. A wide variety of graph problems are connected with these numbers; similar numbers appear for instance in Tutte [9]. An extensive bibliography appears in Brown [3].

The problem of the number of elements in  $S_n$  is a special case of the ballot problem. Elementary discussions of related problems occur in [6] and [10, Problems 54, 83, 84]; a longer discussion and bibliography which includes references to a number of equivalent problems appears in Takács [8].

A number of other structures have also been put on the set  $A$ ; see for instance [7], in which each  $A_n$  is made into a lattice.

**2. Characterizations of the operation with no relations.** Let us regard  $A$  as a set with a binary operation and a generator  $a$ . We observe first that the only automorphism of  $A$  is trivial.

2.1. THEOREM. *Let  $f: A \rightarrow A$  satisfy  $f(bc) = f(b)f(c)$ , for all  $b$  and  $c$  in  $A$ . Then the following are equivalent:*

- (a)  $f$  is onto,
- (b)  $f(a) = a$ ,
- (c)  $f$  is the identity map.

*Proof.* If  $f$  is onto, there is some  $b$  in  $A$  with  $f(b) = a$ . If  $b \neq a$ , then  $b = cd$  for some  $c$  and  $d$  in  $A$ ; hence  $a = f(cd) = f(c)f(d)$ . But since  $a$  cannot be factored, this is impossible, so  $b = a$  as desired. Suppose next that  $f(a) = a$ . Clearly  $f(aa) = f(a)f(a) = aa$ ; also  $f((aa)a) = f(aa)f(a) = (aa)a$ , and by induction  $f(bc) = f(b)f(c) = bc$ , so  $f$  is the identity map  $I: A \rightarrow A$ . Finally,  $I$  is clearly onto.

In 1.3 we defined the “mirror image” map  $M: A \rightarrow A$ . It is clear that  $M(a) = a$  and  $M(bc) = M(c)M(b)$  for all  $b$  and  $c$  in  $A$ .

2.2. THEOREM. Let  $F: A \rightarrow A$  satisfy  $f(bc) = f(c)f(b)$ . Then the following are equivalent:

- (a)  $f$  is onto,
- (c)  $f(a) = a$ ,
- (d)  $f = M$ .

*Proof.* Similar to 2.1.

$M$  is the unique anti-automorphism of  $A$ . Clearly, no other map arising later in the discussion can be an automorphism or anti-automorphism.

A set  $Q$  with a binary operation  $*$  is **isomorphic** to  $A$  if there is a one-one onto map  $f$  from  $Q$  to  $A$  such that  $f(b * c) = f(b)f(c)$  for all  $b$  and  $c$  in  $Q$ . Then  $f$  is an **isomorphism**; a one-one onto map  $g$  such that  $g(b * c) = g(c)g(b)$  will be called an **anti-isomorphism**.

2.3. COROLLARY. Suppose  $(Q, *)$  is isomorphic to  $A$ . Then there is a unique isomorphism  $f: Q \rightarrow A$  and a unique anti-isomorphism  $Mf: Q \rightarrow A$ .

*Proof.* If  $f_1: Q \rightarrow A$  and  $f_2: Q \rightarrow A$  are isomorphisms,  $f_1f_2^{-1}$  and  $f_2f_1^{-1}$  are automorphisms of  $A$ . Hence  $f_1f_2^{-1} = I = f_2f_1^{-1}$  and  $f_2 = f_1$ . Similarly if  $g_1$  and  $g_2$  are anti-isomorphisms,  $g_1g_2^{-1} = I = g_2g_1^{-1}$  and  $g_1 = g_2$ . If an isomorphism  $f$  exists, then  $Mf(bc) = M(f(b)f(c)) = Mf(c)Mf(b)$  so  $Mf$  is the unique anti-isomorphism.

To make it easier to recognize structures isomorphic to  $A$ , we introduce some terminology. If  $Q$  is a set and  $f$  is an operation  $f: Q \times Q \rightarrow Q$ , the structure  $(Q, f)$  will be called **graded** if  $Q = Q_1 \cup Q_2 \cup \dots$  and  $f(Q_n \times Q_m) \subset Q_{n+m}$  for all  $n, m \geq 1$ . If  $Q$  and  $R$  are graded sets, a map  $g: Q \rightarrow R$  will be called **level-preserving** if  $g(Q_n) \subset R_n$  for all  $n$ .

If  $(Q, f)$  is graded,  $Q_1$  is not in the image  $f(Q \times Q)$ . We say **factoring is possible** if  $f: Q \times Q \rightarrow Q \setminus Q_1$  is onto. We say **factoring is unique** if  $f: Q \times Q \rightarrow Q \setminus Q_1$  is both onto and one-one; that is, if each  $q$  not in  $Q_1$  has a unique expression  $q = f(b, c)$ . In particular, factoring in  $A$  is unique.

2.4. THEOREM. Let  $(Q, *)$  be a graded structure in which factoring is unique. Suppose  $Q_1$  has exactly one element. Then there is a level-preserving isomorphism from  $(Q, *)$  to  $A$ .

*Proof.* The unique map  $f: Q_1 \rightarrow A_1$  is one-one onto. If  $f$  has been defined to be one-one, onto, level-preserving, and to preserve multiplication (when possible) from  $Q_1 \cup Q_2 \cup \dots \cup Q_n$  to  $A_1 \cup \dots \cup A_n$ , define  $f$  on  $Q_{n+1}$  by  $f(q) = f(b)f(c)$ , where  $q = b * c$ . This is well-defined since  $Q$  has unique factorization, and  $f(q)$  is in  $A_{n+1}$  since  $f$  is level-preserving on  $b$  and  $c$  by the induction hypothesis. Finally  $f: Q_{n+1} \rightarrow A_{n+1}$  is one-one onto; for if  $d$  is in  $A_{n+1}$ , then  $d$  can be uniquely written  $d = uv$ , and by the induction hypothesis there are unique  $b$  and  $c$  with  $f(b) = u$  and  $f(c) = v$ . Now  $b * c$  is in  $Q_{n+1}$  and is the unique element of  $Q_{n+1}$  mapping to  $uv = d$ .

It would be possible to change the hypotheses of 2.4 so that  $(Q, *)$  is not graded, but  $*: Q \times Q \rightarrow Q \setminus \{a\}$  is one-one, for some  $a$  in  $Q$ . One would then have

to require that  $a$  "generate"  $Q$ , to avoid the possibility of elements of "infinite length."

2.5. COROLLARY. *Let  $(Q, *)$  be a graded structure isomorphic to  $A$ . Then the isomorphism is level-preserving.*

Hence if  $(Q, *)$  is a graded structure isomorphic to  $A$ , the isomorphism is one-one onto between  $Q_n$  and  $A_n$ ; thus  $Q_n$  must have  $a_n$  elements. There is a partial converse to this:

2.6 THEOREM. *Let  $(Q, *)$  be a graded structure in which factoring is possible. Suppose  $Q_n$  has  $a_n$  elements, for all  $n$ . Then factoring is unique and  $(Q, *)$  is isomorphic to  $A$ .*

*Proof.* We must merely observe that for each  $n > 1$ ,

$$*: \cup(Q_k \times Q_{n-k}) \rightarrow Q_n \quad (\text{union for } 1 \leq k \leq n-1)$$

is one-one onto. It is onto by hypothesis, since factoring is possible. However,  $Q_n$  has  $a_n$  elements and the union on the left has

$$\sum a_k a_{n-k} = a_n \quad (\text{sum for } 1 \leq k \leq n-1)$$

elements by the standard recurrence formula for Catalan numbers. Hence to be onto  $Q_n$ , the given map must be one-one. Now since factors of an element of  $Q_n$  must lie in some  $Q_k \times Q_{n-k}$ , factoring is unique.

2.7. Remark. A line of argument comparable to the above is possible for some other combinatorial problems. For instance, let  $B_n$  be the collection of distinct ways of introducing parentheses in a product of  $n$  identical terms under an operation presumed commutative but nonassociative. Let  $b_n$  denote the number of elements of  $B_n$ . It is easy to check that  $b_1 = b_2 = b_3 = 1$  and  $b_4 = 2$  [since  $(xx)(xx) \neq ((xx)x)x$ ]. An element of  $B$  may be factored uniquely "up to commutativity" and thus it may be established that

$$b_n = \sum b_k b_{n-k} + c \quad (\text{sum for } 1 \leq k < n/2),$$

where  $c = 0$  if  $n$  is odd,  $c = \frac{1}{2}b_{n/2}(b_{n/2} + 1)$  if  $n$  is even. Note that if  $M$  were an anti-automorphism of this structure,  $M(cd) = M(d)M(c) = M(c)M(d)$ , so  $M$  would be an automorphism and thus the identity map. If this fact is taken into account, it is easy to rephrase 2.1 through 2.6 for this structure. Some history of these numbers, together with a list of combinatorial problems they solve, appears in Becker [2].

**3. Dissections of a polygon.** We regard  $E_1$  as consisting of a (trivial) 2-gon;  $E_2$  contains the triangle, with no diagonals. By Theorem 2.4, we may provide a one-one onto map  $e: E_n \rightarrow A_n$ , and thus prove that  $E_n$  has  $a_n$  elements, by defining a graded multiplication in  $E$  for which factoring is unique. We give the same operation as in [1], somewhat differently phrased. Let  $B$  and  $C$  denote

two dissections of an  $(n+1)$ -gon and an  $(m+1)$ -gon respectively; consider each to have one edge marked as base. Take another edge  $N$  in the plane; translate  $B, C$ , and the new edge to form a triangle, whose sides in clockwise order are  $(N)$  (base of  $B$ ) (base of  $C$ ). The result is combinatorially a triangulated  $(n+m+1)$ -gon with base  $N$ . Figure 1 illustrates a typical  $B, C$ , and their product  $B \cdot C$ .

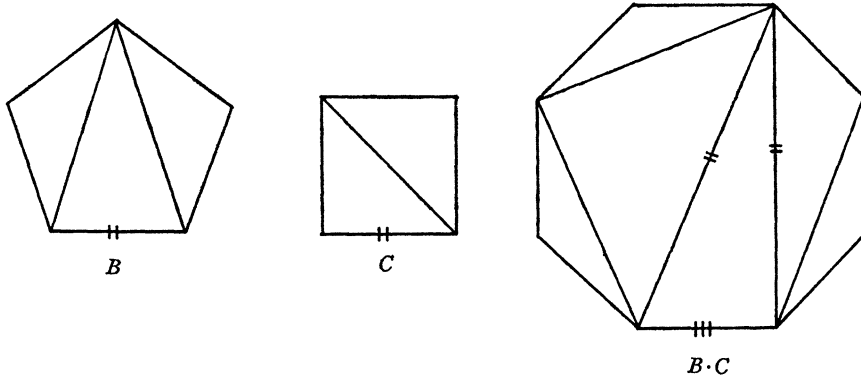


FIG. 1

We now observe that this product, denoted by  $\cdot$  (usually omitted), has unique factorization. Given an element of  $E_n, n > 1$ , with a base specified, the base belongs to a unique triangle. Erase the base and let the other two sides of its triangle be the bases for their respective subpolygons. Thus Fig. 1.  $B$  is the product of two triangles; Fig. 1.  $C$  is the product of a trivial 2-gon by a triangle; any polygon may be factored uniquely except the trivial 2-gon of  $E_1$ .

By Theorem 2.4, there is now a unique isomorphism  $e: E \rightarrow A$ . We describe it as follows: beginning at the center of the base, proceed clockwise around the perimeter. Write down a parenthesis “(” whenever you pass an end of some diagonal whose other end you have not yet passed. Write down an “a” whenever you pass a midpoint of any side except the base. Write down a parenthesis “)” whenever you pass an end of a diagonal whose other end you have passed previously. Thus Fig. 1.  $B \cdot C$  maps to  $((aa) (aa)) (a(aa))$  while  $B$  and  $C$  map to  $(aa) (aa)$  and  $a(aa)$  respectively.

Let  $\hat{M}$  denote reflection in the perpendicular bisector of the base, and  $\rho_n$  denote counterclockwise rotation through  $2\pi/(n+1)$  radians. Now  $\rho: E \rightarrow E$  may be defined by  $\rho(B) = \rho_n(B)$  for  $B$  in  $E_n$ . Let  $I: E \rightarrow E$  denote the identity map. The following lemmas are immediate.

3.1. LEMMA. *The relations  $\hat{M}^2 = I, \rho_n^{n+1} = I$ , and  $\hat{M}\rho = \rho^{-1}\hat{M}$  hold.*

3.2. LEMMA. *The order of  $\rho_n$  is 1 for  $n = 1$  and 2, is 2 for  $n = 3$ , and is  $n + 1$  for  $n > 3$ . Hence  $\rho$  has infinite order.*

*Proof.* The dissection  $D$  of the  $(n+1)$ -gon in which all  $n-2$  diagonals begin at the left end of the base [hence,  $e^{-1}(((aa)a \cdots )a)$ ] has for  $n > 3$  the property

that  $D, \rho_n D, \rho_n^2 D, \dots, \rho_n^n D$  are distinct. Hence the order of  $\rho_n \geq n+1$  for  $n > 3$ . The upper bound is given in 3.1, and the cases  $n = 1, 2, 3$  follow by inspection.

We are now able to observe that  $\rho_n, n > 3$ , cannot preserve any reasonable graded multiplication on  $E$ .

3.3. THEOREM. *Let  $*$ :  $E \times E \rightarrow E$  be any graded multiplication for which factoring is possible. Let  $\tau$ :  $E \rightarrow E$  be any level-preserving map such that  $\tau|_{E_k} = \rho_k$  for at least one  $k \geq 4$ . Then neither  $\tau(B * C) = \tau(B) * \tau(C)$  nor  $\tau(B * C) = \tau(C) * \tau(B)$  can hold for all  $B$  and  $C$  in  $E$ .*

*Proof.* By Theorem 2.6,  $(E, *)$  is isomorphic to  $A$ . Thus  $\tau$  induces a level-preserving map  $\tau_A$  of  $A$  whose order is not less than  $k+1$ . Hence  $\tau_A$  is not  $I$  or  $M$ , and cannot preserve multiplication.

Let  $(E, \cdot)$  be the previously defined binary structure on  $E$ , and  $e$ :  $(E, \cdot) \rightarrow A$  the unique isomorphism. It is clear by inspection that  $\hat{M}(B \cdot C) = \hat{M}(C) \cdot \hat{M}(B)$ ; hence  $\hat{M}$  is the unique anti-automorphism of  $(E, \cdot)$ . Now  $e\hat{M}e^{-1}$  is an anti-automorphism of  $A$ , so  $e\hat{M}e^{-1} = M$ . Let  $\rho_A$  denote  $e\rho_n e^{-1}$ .

3.4. PROPOSITION: *The relations  $M^2 = I$  and  $M\rho_A = \rho_A^{-1}M$  hold. Also  $\rho_A|_{A_n}$  has order  $n+1$  for  $n \geq 4$  and  $\rho_A: A \rightarrow A$  has infinite order.*

*Proof.* Immediate; for  $M\rho_A = e\hat{M}e^{-1}e\rho_n e^{-1} = e\hat{M}\rho_n e^{-1} = e\rho_n^{-1}\hat{M}e^{-1} = \rho_A^{-1}M$ , and similarly for the other statements.

Since we cannot hope to find maps of  $A$  which preserve multiplication, the collection of maps which anti-commute with  $M$  (i.e., maps  $\tau: A_n \rightarrow A_n$  such that  $M\tau = \tau^{-1}M$ ) may appear worthwhile to study. We tentatively call such maps **rotations**. Thus  $\rho_A$  is a rotation of infinite order on  $A$  and of order  $n+1$  on  $A_n$  for  $n \geq 4$ .

We illustrate with a rotation of order 2, i.e., a map  $\beta: A \rightarrow A$  such that  $\beta M = M\beta$ . Define  $\beta: A \rightarrow A$  by the rules  $\beta(a) = a$  and  $\beta(bc) = M(\beta(b)\beta(c))$ .

3.5. PROPOSITION. *We have  $\beta^2 = M^2 = (M\beta)^2 = I$ , so  $\{I, M, \beta, M\beta\}$  is the noncyclic four-group.*

*Proof.* Clearly  $\beta(a) = M(a) = (M\beta)^2(a) = a$ , so the relations desired hold on  $A_1$ . Suppose they hold on  $A_k$ , for  $k \leq n$ , and let  $bc$  be in  $A_{n+1}$ . Now

$$\begin{aligned} \beta^2(bc) &= \beta M(\beta(b)\beta(c)) = \beta(M\beta(c)M\beta(b)) \\ &= M(\beta M\beta(c)\beta M\beta(b)) = M\beta M\beta(b)M\beta M\beta(c) \end{aligned}$$

which is  $bc$  by the induction hypothesis that  $M\beta M\beta = I$ . Similarly

$$\begin{aligned} M\beta M\beta(bc) &= M\beta M[M(\beta(b)\beta(c))] = M\beta(\beta(b)\beta(c)) \\ &= MM(\beta^2(b)\beta^2(c)) = bc, \end{aligned}$$

by the hypothesis  $\beta^2 = I$ .

We observe that  $I, M, \beta, M\beta$  are distinct since  $((aa)a)a$  is carried by them to  $((aa)a)a, a((aa)), a((aa)a)$ , and  $(a(aa))a$  respectively.

4. Sequences of plus and minus. Given that  $S_n$  has the same number of



elements as  $A_n$ , which we shall prove shortly, we have the following surprising result:

4.1. THEOREM. *Let  $*$  be any graded multiplication on  $S$  for which factoring is possible. Then  $\lambda: S \rightarrow S$  does not preserve multiplication.*

*Proof.* By Theorem 2.6,  $(S, *)$  is isomorphic to  $A$ . By examples in 1.4 and 1.3 respectively,  $\lambda$  is carried to neither  $M$  nor  $I$ . Hence  $\lambda$  is a map from  $(S, *)$  onto  $(S, *)$  which is neither an automorphism nor an anti-automorphism.

We now introduce a graded multiplication in  $S$ , making it isomorphic to  $A$  and thus establishing that  $S_n$  has the same number of elements as  $A_n$ . Note that  $S_1$  consists of the empty sequence  $( )$  and  $S_2$  of the single sequence  $(+1, -1)$ . If  $s_n$  is in  $S_n$  and  $s_m$  is in  $S_m$ , define  $s_n * s_m$  to be  $(+1, s_n, -1, s_m)$ . For instance

$$(+1, -1, +1, -1) * (+1, -1) = (+1, +1, -1, +1, -1, -1, +1, -1).$$

If  $s_n$  has  $2n - 2$  terms and  $s_m$  has  $2m - 2$  terms, then  $s_n * s_m$  has  $2(n + m) - 2$  terms as desired.

We now establish that factoring is unique in  $(S, *)$ . Suppose  $(x_1, x_2, \dots, x_{2n-2})$  is in  $S_n$ . Let  $2k$  be the unique integer between 2 and  $2n - 2$  such that  $x_1 + x_2 + \dots + x_{2k} = 0$  and  $x_1 + x_2 + \dots + x_j > 0$  for  $j < 2k$ . Then  $x_{2k} = -1$  and

$$(x_1, x_2, \dots, x_{2n-2}) = (x_2, \dots, x_{2k-1}) * (x_{2k+1}, \dots, x_{2n-2}),$$

where the first factor is in  $S_k$  and the second in  $S_{n-k}$ . This way of factoring is the only one, since if

$$\begin{aligned} (+1, x_2, \dots, x_{2k-1}, -1, x_{2k+1}, \dots, x_{2n-2}) \\ = (x_2, \dots, x_{2k-1}) * (x_{2k+1}, \dots, x_{2n-2}), \end{aligned}$$

then we must have  $x_1 + x_2 + \dots + x_{2k} = +1 + (x_2 + \dots + x_{2k-1}) - 1 = +1 + 0 - 1 = 0$  and  $x_1 + x_2 + \dots + x_j = +1 + (x_2 + \dots + x_j) \geq +1 + 0 > 0$  for  $j < 2k$ .

This definition of multiplication seems noticeably skewed. It is reasonable to introduce another one:

$$s_n \circ s_m = (s_n, +1, s_m, -1).$$

The same considerations as for  $*$  apply, and we have the relations

$$\begin{aligned} \lambda(s_n \circ s_m) &= \lambda(s_n, +1, s_m, -1) = (+1, \lambda s_m, -1, \lambda s_n) \\ &= \lambda s_m * \lambda s_n, \\ \lambda(s_n * s_m) &= \dots = \lambda s_m \circ \lambda s_n. \end{aligned}$$

Let  $f_*: (S, *) \rightarrow A$  and  $f_o: (S, \circ) \rightarrow A$  denote the unique isomorphism from  $(S, *)$  and  $(S, \circ)$  respectively to  $A$ . Let  $\sigma$  denote  $f_o^{-1} f_*: (S, *) \rightarrow (S, \circ)$ .

4.2. THEOREM. *The mapping  $\sigma$  is the unique isomorphism from  $(S, *)$  to  $(S, \circ)$ ;  $\lambda$  is the unique anti-isomorphism from  $(S, \circ)$  to  $(S, *)$ . Hence  $\lambda \sigma$  and  $\sigma \lambda$  are the unique anti-automorphisms of  $(S, *)$  and  $(S, \circ)$  respectively.*

*Proof.* Immediate from the uniqueness theorems of Section 2 and the formula  $\lambda(s_n \circ s_m) = \lambda s_m * \lambda s_n$ .

Since  $\lambda\sigma$  is an anti-automorphism,  $(\lambda\sigma)^2 = 1$ . Hence  $\lambda\sigma = \sigma^{-1}\lambda$  and  $\lambda\sigma^{-1} = \sigma\lambda$ . It also follows that  $\sigma$  is a rotation of  $(S, *)$ , as we have defined that term; for the anti-automorphism of  $(S, *)$  is  $\lambda\sigma$  and  $(\lambda\sigma)\sigma = \sigma^{-1}(\lambda\sigma)$  as required.

By contrast,  $(\lambda\sigma)\lambda = \sigma^{-1} \neq \sigma = \lambda(\lambda\sigma)$ , so  $\lambda$  is not a rotation by our definition. This may, however, indicate a failure of the definition, for this is *not* an intrinsic property of  $\lambda$ .

4.3. PROPOSITION. *There exists a binary operation  $f: S \times S \rightarrow S$  such that  $(S, f)$  is isomorphic with  $A$ , and if  $\hat{M}: (S, f) \rightarrow (S, f)$  is the unique anti-automorphism, then  $\lambda\hat{M} = \hat{M}\lambda$ .*

*Proof.* We must merely show that such an operation  $f$  can be chosen, out of the very large number of operations on  $S$  for which factoring is unique. We shall first describe a map  $\hat{M}: S \rightarrow S$ .

STEP 1. Let  $\hat{M}(s) = \lambda(s)$  for all  $s$  in  $S$  for which  $\lambda(s) \neq s$ .

STEP 2. If  $n$  is odd,  $S_n$  has an even number of elements by recurrence (2) of Section 1. Organize those not used in Step 1 into pairs  $(s, t)$ , and let  $\hat{M}(s) = t$  and  $\hat{M}(t) = s$ .

STEP 3. In  $S_{2n}$ , choose arbitrarily  $a_n$  elements  $s$  for which  $\lambda(s) = s$ , and for them let  $\hat{M}(s) = s$ . At least  $a_n$  such elements exist, since if  $t$  is in  $S_n$ , then  $(t, +1, -1, \lambda t)$  is such an  $s$ . However,  $a_{2n} - a_n$  is even (since  $a_n^2 - a_n$  is even), so the remaining elements for which  $\lambda(s) = s$  may be paired as in Step 2.

We now describe an operation  $f: S \times S \rightarrow S$ :

STEP 4. For  $s$  in  $S_n$ , let  $f(s, \hat{M}(s))$  be an element  $t$  of  $S_{2n}$  with  $\hat{M}(t) = t$ . This can be done in a one-one onto fashion since there are  $a_n$  pairs  $(s, \hat{M}(s))$ .

STEP 5. Each other pair  $(s, t)$  in  $S \times S$  has an associated pair  $(\hat{M}(t), \hat{M}(s))$  distinct from itself. Map  $(s, t)$  and  $(\hat{M}(t), \hat{M}(s))$  to an arbitrarily chosen pair  $f(s, t)$  and  $\hat{M}(f(s, t))$  of suitable grade.

It is now easy to check that  $\hat{M}$  is the unique anti-automorphism of  $(S, f)$  and that  $\lambda\hat{M} = \hat{M}\lambda$ .

We examine briefly the map in  $A$  corresponding to  $\sigma$ ; let  $\mu = f_* f_{\circ}^{-1}: A \rightarrow A$ .

4.4. PROPOSITION. *We have  $M\mu M\mu = I: C \rightarrow C$ ; hence  $\mu M = M\mu^{-1}$ .*

*Proof.* Clearly  $M = f_* \lambda \sigma f_*^{-1}$ , since  $f_* \lambda \sigma f_*^{-1}$  is an anti-automorphism of  $A$ . Since  $\mu = f_* \sigma f_*^{-1}$ , we have  $M\mu M\mu = f_* \lambda^2 \sigma^{-2} \sigma^2 f_*^{-1} = I$ .

Hence  $\mu$  is a rotation of  $A$ . By observation of its behaviour on  $A_6$ , this rotation must have order at least 30; it is almost certainly infinite. We observe here that neither  $\mu$  nor  $\rho_A$  is a power of the other, since

$$\mu(((aa)a)a) = f_* f_{\circ}^{-1} (((aa)a)a) = f_*(+1, -1, +1, -1, +1, -1) = a(a(aa))$$

and  $\mu(a(a(aa))) = ((aa)a)a$ , but repeated application of  $\rho_A|A_4$  to  $((aa)a)a$  yields in turn all five elements of  $A_4$ , since the five Euler triangulations of the pentagon are all rotated images of each other (one appears in Fig. 1.B).

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## ARZELÀ'S DOMINATED CONVERGENCE THEOREM FOR THE RIEMANN INTEGRAL

W. A. J. LUXEMBURG, California Institute of Technology

**1. Introduction.** Riemann's definition ([14], p. 239) of a definite integral gave rise to a number of important developments in analysis. In the course of these developments a remarkable result due to C. Arzelà ([1], 1885) marked the beginning of a deeper understanding of the continuity properties of the Riemann integral as a function of its integrand. The result of Arzelà we have in mind is the so-called ARZELÀ DOMINATED CONVERGENCE THEOREM for the Riemann integral concerning the passage of the limit under the integral sign. It reads as follows.

**THEOREM A** (C. Arzelà, 1885). *Let  $\{f_n\}$  be a sequence of Riemann-integrable functions defined on a bounded and closed interval  $[a, b]$ , which converges on  $[a, b]$  to a Riemann-integrable function  $f$ . If there exists a constant  $M > 0$  satisfying  $|f_n(x)| \leq M$  for all  $x \in [a, b]$  and for all  $n$ , then  $\lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)| dx = 0$ . In particular,*

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx = \int_a^b f(x) dx.$$

Usually, Arzelà's theorem is formulated as a result about term-by-term

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Professor Luxemburg received his Ph.D. under A. C. Zaanan at the Technological Univ. of Delft. He spent a year on a Canadian National Research Council Fellowship, two years at the Univ. of Toronto, and joined Cal. Tech. in 1958, where he now serves as the Math. Dept. Chairman. He spent a half year leave at the Univ. of Leiden in 1965. His main research interests are classical, functional, and non-standard analysis. He edited *Applications of Model Theory* (Holt, Rinehart and Winston, 1969). *Editor.*