$$\left\{\int_a^x \left[(1-\alpha)f(t)\exp\left(\int_a^t (1-\alpha)b(\tau)d\tau\right)\right] dt + c^{1-\alpha}\right\}^{1/(1-\alpha)}.$$

Also solved by J. E. Chance, F. A. Homann, S. J., A. A. Jagers (Netherlands), G. A. Kemper, Charlotte Krauthammer (Austria), J. R. Kuttler, Beatriz Margolis (Argentina), R. J. Schaar, J. S. Shipman, T. Teichmann, H. C. Wente, and the proposer.

The original statement contained a misprint, as discovered by all contributors: as first printed, $b(\tau)$ had an incorrect coefficient $(1-\alpha)$.

Discontinuities of Functions in $\mathbb{R}^2$

5844 [1972, 307]. *Proposed by L.-S. Hahn, University of New Mexico*

Construct a function defined everywhere in the plane which is nowhere continuous and yet is continuous in each variable separately, or prove such a function does not exist.

Solution by G. M. Leibowitz, University of Connecticut. In volume one of E. W. Hobson, *The Theory of Functions of a Real Variable*, reprinted by Dover, 1957, we see on p. 449 that if f is separately continuous in each variable, then f is continuous at points on each graph of a continuous function. Hence no such function exists.

Also solved by Bruce Ferrero, O. P. Lossers (Netherlands), C. J. Neugebauer, T. Šalát (Czechoslovakia), and the proposer.

Note. We are referred by Šalát to F. W. Carroll, *Separately continuous functions*, this MONTHLY, V. 78 (1971), p. 175; and by Lossers to C. Goffman, *Real Functions*. The critical fact is that f is in the first Baire class.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

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All unsigned material is written by the editors. A boldface capital **C** *in the margin indicates that a review is based in part on classroom use. Professors willing to write such a review should inform the editor in order to avoid duplication.*

Perspectives in Mathematics. By David E. Penney. Benjamin, Menlo Park, California, 1972. xiv + 349 pp. $9.95. (Telegraphic Review, April 1972.)

Mathematics in Civilization. By H. L. Resnikoff and R. O. Wells, Jr. Holt, Rinehart, and Winston, New York, 1971. viii + 583 pp. Preliminary Edition. (Telegraphic Review, January 1972.)

Liberal Arts Mathematics. By Lawrence Spector. Addison-Wesley, Reading, Massachusetts, 1971. xv + 560 pp. $9.95. (Telegraphic Review, April 1972.)

Mathematics: A Humanistic Approach. By G. Joseph Wimbish. Wadsworth, Belmont, California, 1972. 169 pp. $3.50 (Paper).

Readings for Mathematics: A Humanistic Approach. 191 pp. $4.95 (Paper).

It has long been common for college undergraduates in areas outside mathematics, science, or engineering to be subjected to some elementary mathematics course other than calculus — most typically covering some form of logic, probability, or finite mathematics. In the last few years there has been a major growth of alternatives to these courses. Titles seem to range from "Introduction to Mathematics" and "Mathematics for the Liberal Arts" to "Mathematics for Poets," a title rarely seen in catalogues but perhaps the name most commonly used by mathematicians in casual discussion. The four books under review are only a few of the many published recently for such courses. They differ dramatically from one another; if any common feature exists, it is that they are fairly extreme examples of the different types of books now available.

Just whom are such courses and texts aimed at? Very often such a course is intended to siphon nontechnical students out of previously required courses such as calculus, in the hope of leaving the calculus instructors a better-motivated group of students. Usually there is some additional hope that these courses may be more interesting than calculus, and thus draw people toward mathematics. Perhaps on rare occasions such a course is introduced out of a real understanding of a liberal arts education, by someone who thinks diversification courses should be liberal (i.e., liberating) and feels that many mathematics courses lack something in this regard.

Many teachers of undergraduates certainly desire that undergraduates at least have the opportunity to discover what mathematics is, what mathematicians do, and why. Remarkably few people, for instance, have any notion of what a mathematician does for "research." Calculus courses have a strong tendency to be unhelpful in this regard, and other diversification courses have usually been little better.

Ideally, a diversification course might include some material in each of these general areas: (1) The internal structure of mathematics. The idea that there are postulates and theorems, and perhaps the notion that the postulates may *not* be simple statements about the real world, but chosen for another reason. (2) The history of mathematics — i.e., how did the strange way of doing business described in (1) develop? My own experience is that students tend to demand more historical comment than most mathematicians expect. (3) Some nontrivial applications. This probably (but not invariably) means nonnumerical applications, and it would be nice to use these in response to the ubiquitous "what is it good for" questions.

In addition to these three areas, many teachers want to give some sense of aesthetics — i.e., enough theory to allow the students to see a "pretty" theorem. On the other hand, these new courses are often not intended to give computational expertise in the way traditional courses are. Indeed, most such courses are not prerequisites to specific courses in other departments or even to other mathematics courses.

The four books under consideration here represent a broad range of possible courses. We describe each of them briefly before returning to broad remarks; it should be noted that our choice of features in each book to comment on is based principally on the sort of elementary course described above rather than on other uses the books may have.

Penney contains ten essentially independent chapters, each of which is a fairly detailed examination of a specific topic. The mathematics involved is often quite deep, and the book is an excellent reference for anyone teaching an elementary topics course. At the same time, I suspect it would frighten off all but the brightest undergraduates; it might be more appropriate to advanced mathematics majors than to freshmen, and it contains a wealth of material for, e.g., honors seminars. For instance, the chapter on infinite sets contains a proof of the Cantor-Schroeder-Bernstein Theorem; the chapter on animal populations, in its discussion of predator-prey differential equations, approaches the area of currently published research in biochemistry. There is a detailed discussion of logarithms, continued functions, and piano tuning; a chapter on combinatorial topology (including Steinitz's theorem and the 5- and 7-color theorems); chapters on group theory, number theory, topics in geometry, real numbers. There are numerous exercises (with brief hints in the back), some of which are worked into the text as part of the main chain of thought. There are excellent bibliographies and an index.

All in all, Penney contains a wealth of material and is worth owning, especially if your curriculum includes an elementary topics course; but it is more useful for advanced students than as a text in a multi-sectioned diversification course, and will hardly substitute for e.g., Beck *et al* **[1]** or the more elementary Youse **[6]** for the latter use. It could well be used in places where Rademacher and Toeplitz **[5]** is tempting; the more sustained arguments, and exercises, are a real advantage.

Resnikoff and Wells is organized on an historical basis, giving historical excerpts from early Egyptian and Babylonian arithmetic up through the development of calculus. The three largest divisions of the text cover (I) earliest developments of the notion of number and land measurement; (II) trigonometry, navigation, cartography, logarithms; and (III) calculus and differential geometry. The last chapter, "Models of the Universe," gets into a discussion of relativity, particularly the question of the shape of the universe. The historical discussions are very readable and well thought out to motivate the mathematics; such mathematics as is necessary to follow the text should be readily comprehensible by undergraduates. The use of questions of navigation (e.g., Columbus) to motivate trigonometry and cartography is exceptionally well done. The quantity of material given is quite appropriate for a year course. On the other hand, the pure mathematician may feel that the axiomatic method is not sufficiently stressed and that the material is too closely restricted to numerical questions to give a fair view of what mathematicians actually do.

There are a moderate variety of computational exercises, advancing to where they give a fair indication of what calculus is like (simple integrals, partial derivatives, Taylor series). Many chapters also have historical questions asking why a given idea seems significant, and other questions calling for narrative rather than computational answers.

Spector is noticeably more elementary than the other books under consideration. According to the preface it "acknowledges only the barest absorption from high school" and is rooted "in arithmetic and not in analysis." The first five chapters cover set theory and logic (Venn diagrams and truth tables, for instance). The next four cover arithmetic, including the basic material customary in courses for elementary education majors (arithmetic in other bases, repeating and nonrepeating decimals). Chapter 10 gives the axioms for a field (stated for the reals) and states Gödel's incompleteness result; Chapter 11 contains a brief introduction to infinity (but not, for instance, a valid proof that the reals are uncountable). Chapter 12 is a brief discussion of probability. The author reports that the first twelve chapters can be covered reasonably in one semester. The remaining six chapters constitute an elementary introduction to number theory.

One very strong feature of this text is the inclusion of numerous exercises with complete answers and explanations provided. This makes it usable for independent study and minimizes the need for in-class review of homework. The book could be highly appropriate for a course with very weak students or in a course that must include elementary education majors with other students. Most mathematicians may find it too elementary to be satisfying for the sort of course under specific consideration here. At the same time, in a class of very mixed ability, it might be worth considering using a book as self-explanatory as this one in order to free class time for lectures going somewhat deeper.

Wimbish is highly unusual, both in format and contents. It comes in two paperbacks, a text and an associated book of readings. The text is very informal, and often one gets the feeling of Wimbish informally philosophising in front of a class. A few areas have a small amount of concrete content: axiomatics, truth tables, elementary syntax, computer programming. An appendix contains a bit more set theory, but still not much. Some sections have exercises; no answers are included. There is an index, and scattered references to other books appear in footnotes. There are quite a number of errors, which a student would find very confusing. For instance, the first set of axioms for which the reader is asked to prove a theorem, on pages 43–44, contains

> Axiom C5: For every piece of wood, there exists at least one other piece of wood that can be joined to it by hammering.

The axiom follows from the previous ones: Theorem C1 which should be numbered C6 and which the reader is asked to prove, does not.

The readings volume contains a baker's dozen of items — from excerpts from Plato, Kant, and Lewis Carroll through semipopular articles by Halmos, Kline, and Quine — including adequate material to dramatically supplement the text volume. Yet even those articles that could be used to motivate computation are followed by class-discussion rather than more mathematical questions.

Despite imperfections, the books by Wimbish provide a very attractive new approach to textbooks in this area. The first printing is reportedly small enough that we can hope for reasonably prompt improvements in the text. The physical size of the books, and readability of much of both volumes, could avoid "turning off" students who dislike standard mathematics texts. The readings might be used alone or as a supplement to some other book. The articles are deeper mathematically than, say, those in Fadiman [**2**], and the volume is less expensive (and less overwhelming) than Kline [**4**].

To return to broad questions, I would like to comment on the relation of these books to a course I recently taught of the type under consideration here. I used a number of comparatively inexpensive paperbacks, tied together by lectures and dittoed sheets. Rademacher and Toeplitz [**5**] proved too difficult; Fadiman [**2**] and Gamow [**3**] were universally well received; the students repeatedly asked more historical questions than I anticipated. I think this class would have found Penney too hard, and Spector too easy (but they took the course as an elective, so were perhaps somewhat better than average). Wimbish is not too far from the spirit of what I did (although my topics were closer to Penney's) but at least some of the students would have preferred Resnikoff and Wells.

The moral may be that in planning a "liberating" course, a faculty should realize the huge variety of approaches now available. Perhaps the day is at hand when an individual instructor should be allowed to pick a book whose approach he likes, rather than having departments require a standard syllabus in many sections of a course. I personally will be tempted to continue using paperbacks (including perhaps Wimbish's readings) partly because of one thrill (due to Fadiman in this case) that comes rarely to mathematics teachers: having students turn up wanting to take a course because "I read my roommate's math book and it looked like fun."

References

1. A. Beck, M. Bleicher, and D. Crowe, Excursions into Mathematics, Worth Publishers, New York, 1969. $10.75. Reviewed, this MONTHLY, February 1972. pp. 193–194.

2. C. Fadiman, Fantasia Mathematica, Simon and Schuster, New York, 1958. $1.95 (Paper).

3. G. Gamow, One Two Three — Infinity, Bantam Books, New York, (1971 printing), $1.25 (Paper).

4. M. Kline (ed.) Mathematics in the Modern World: Readings from Scientific Amercian, W. H. Freeman, San Francisco, 1968. $6.50 (Paper).

5. H. Rademacher and O. Toeplitz, The Enjoyment of Mathematics, Princeton University Press, Princeton, 1957. $1.95 (Paper).

6. B. Youse, An Introduction to Mathematics, Allyn and Bacon, Boston, 1970. $9.95.

EDWARD T. ORDMAN, University of Kentucky

Free Rings and Their Relations. By P. M. Cohn. Academic Press, New York, 1971. xvi + 346 pp. $22.00. (Telegraphic Review, June-July, 1972.)

In a basic paper in 1964 P. M. Cohn defined the concept of a right fir or free ideal ring: a ring (with 1) in which all right ideals are free and of unique rank. The ensuing years have seen a series of papers, mostly by Cohn and G. M. Bergman, which have developed this field. It is especially valuable to have these results, some previously unpublished results from Bergman's 1967 Harvard thesis, as well as new results, collected together in a book whose prerequisites are quite minimal (apart from the usual "mathematical maturity" presupposed).

Before proceeding to a more detailed account of the book, one might remark that its object is the study of free associative algebras and related rings. The study of such algebras, which is essentially the study of non-commutative polynomial rings over skew fields, will hopefully in the future shed some light on solutions of algebraic equations in non-commuting indeterminants with coefficients in skew fields. This in turn would be the basis of non-commutative algebraic geometry.

As mentioned above, the actual prerequisites needed to read the book are minimal — a first year graduate algebra course should suffice. Chapter 0 contains all the necessary material which one might not see in such a course, such as eigenrings, centralizers, Öre rings, free associative algebras, and skew polynomial rings.

In Chapter 1 firs and their various generalizations are defined and studied. Chapter 2 is devoted to the study of generalizations and analogues of the Euclidean algorithms. In Chapter 3 commutative unique factorization is reviewed, before passing to the non-commutative case. Chapter 4 examines rings with distributive factor lattices. Torsion modules over firs and semi-firs are the subject of Chapter 5. Subrings of firs are studied in Chapter 6. Chapter 7 contains some of the author's results on a topic of long-standing interest: skew fields of fractions. The final Chapter (8) is devoted to the study of the additional information available when we have a principal ideal domain. There are two appendices containing basic results on (1) lattice theory, and (2) categories and homological algebra.

The book has two additional features which add to its value: (1) a large number of exercises following each section, and (2) notes and (historical) comments at the end of each chapter.

It remains only to say that this excellent book will probably long remain a basic reference text for anyone seriously interested in the field.

D. J. FIELDHOUSE, Kingston, Ontario, Canada