



---

Sequential Conditions and Free Topological Groups

Author(s): Edward T. Ordman and Barbara V. Smith-Thomas

Source: *Proceedings of the American Mathematical Society*, Vol. 79, No. 2 (Jun., 1980), pp. 319-326

Published by: American Mathematical Society

Stable URL: <http://www.jstor.org/stable/2043259>

Accessed: 29/04/2010 16:34

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=ams>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).



American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings of the American Mathematical Society*.

<http://www.jstor.org>

## SEQUENTIAL CONDITIONS AND FREE TOPOLOGICAL GROUPS

EDWARD T. ORDMAN AND BARBARA V. SMITH-THOMAS

**ABSTRACT.** Most of the results in this paper concern relationships between sequential properties of a pointed topological space  $(X, p)$  and sequential properties of the Graev free topological group on  $X$ . In particular, it is shown that the free group over a sequential  $k_\omega$ -space is sequential, and that a nondiscrete sequential free group has sequential order equal to  $\omega_1$  (the first uncountable ordinal). The free topological group on a space  $X$  which includes a convergent sequence contains a closed subspace homeomorphic to  $S_\omega$ , a previously studied homogeneous, zero-dimensional sequential space. Finally, it is shown that there is no topological group homeomorphic to  $S_\omega$ .

**0. Introduction.** In this paper we discuss relationships between sequential properties of a pointed topological space  $(X, p)$  and sequential properties of its Graev free topological group  $F_G(X, p)$ . Sequential spaces have been discussed in [D], [Fr<sub>1</sub>], [Fr<sub>2</sub>], [A-Fr], [R]; we make heavy use of the space  $S_\omega$  of sequential order  $\omega_1$  constructed in [A-Fr]. The Graev free topological group  $F_G(X, p)$  on a pointed Tychonoff space  $(X, p)$  was introduced in [Gr<sub>1</sub>]; its topology has proved to be rather intractable, but in the last few years good results have been obtained in the case when  $X$  is a  $k_\omega$ -space, that is, a weak union of countably many compact subsets [O<sub>1</sub>], [H-M], [MMO].

Definitions and preliminary results about sequential spaces appear in §1; §2 contains preliminaries about free topological groups and about  $k_\omega$ -spaces. §3 contains results about sequential properties of free topological groups and their consequences. The result that  $S_\omega$  supports no group structure (answering in the negative a question of S. P. Franklin) appears in §4.

We thank S. P. Franklin for posing to us the question just mentioned, and for several helpful conversations and suggestions.

**1. Sequential spaces.** Our definition of sequential spaces, sequential order, and the particular example  $S_\omega$ , are based on [A-Fr].

A subset  $U$  of a topological space  $X$  is *sequentially open* if each sequence converging to a point in  $U$  is eventually in  $U$ . The space  $X$  is *sequential* if each sequentially open subset of  $X$  is open. For each subset  $A$  of  $X$ , let  $s(A)$  denote the set of all limits of sequences of points of  $A$ .  $X$  is of *sequential order 1* ( $X$  is also called a *Fréchet space*) if  $s(A)$  is the closure of  $A$  for every  $A$ .

---

Presented to the Society, January 27, 1977; received by the editors February 5, 1979 and, in revised form, June 4, 1979.

AMS (MOS) subject classifications (1970). Primary 54D55, 22A99, 20E05.

*Key words and phrases.* Sequential space, sequential order,  $k_\omega$ -space, free topological group, Graev free topological group, sequential coreflection.

© 1980 American Mathematical Society  
0002-9939/80/0000-0284/\$03.00

We define higher sequential orders by induction. Let  $s_0(A) = A$ , and for each ordinal  $\alpha = \beta + 1$ , let  $s_\alpha(A) = s(s_\beta(A))$ . If  $\alpha$  is a limit ordinal, let  $s_\alpha(A) = \bigcup \{s_\beta(A) \mid \beta < \alpha\}$ . The sequential order of a sequential space  $X$  is the least ordinal  $\alpha$  such that  $s_\alpha(A)$  is the closure of  $A$  for every subset  $A$  of  $X$ ; this order always exists and does not exceed the first uncountable ordinal  $\omega_1$ .

By  $(S_1, s_0)$  we mean a space consisting of a single convergent sequence  $s_1, s_2, s_3, \dots$ , together with its limit point  $s_0$  taken as the basepoint.

$(S_2, s_0)$  is a space obtained from  $S_1$  by attaching to each isolated point  $s_n$  of  $S_1$  a sequence  $s_{n,1}, s_{n,2}, s_{n,3}, \dots$  converging to  $s_n$ .  $S_2$  may be viewed as a quotient of a disjoint union of convergent sequences; we give it the quotient topology. Inductively, we obtain the space  $(S_{n+1}, s_0)$  from  $(S_n, s_0)$  by attaching a convergent sequence to each isolated point of  $(S_n, s_0)$  and giving the resulting set the quotient topology.

Finally, let  $(S_\omega, s_0)$  be the union of the sets  $S_1 \subset S_2 \subset S_3 \subset \dots$  with the weak union topology (a subset of  $S_\omega$  is closed if and only if its intersection with each  $S_n$  is closed in the topology of  $S_n$ ).

We shall use heavily the following facts from [Fr<sub>1</sub>], [Fr<sub>2</sub>], and [A-Fr]:  $S_n$  is sequential of order  $n$ ;  $S_\omega$  is sequential of order  $\omega_1$  and is countable, homogeneous, and zero-dimensional. A closed subspace of a sequential space (of order  $\alpha$ ) is sequential (of order  $\leq \alpha$ ). A quotient of a sequential space is sequential. The pointed union (one-point union) of sequential spaces is sequential. The weak union of a nest of sequential spaces is sequential.

By a theorem of Boehme [B], the cartesian product of a sequential space and a locally compact sequential space is again sequential. The local compactness condition cannot be entirely removed: the cartesian product of two sequential spaces (even of a metric space and a sequential space) need not be sequential. Let  $(\mathcal{Q}, p)$  denote the rationals with basepoint  $p = 0$ ; let  $(W, q)$  denote the union of countably many copies of the unit interval  $[0, 1]$  with all the basepoints  $q = 0$  identified and with the quotient topology. Then  $\mathcal{Q}$  is a sequential (in fact, a metric) space and  $W$  is sequential, but  $(\mathcal{Q} \times W, (p, q))$  is not sequential. To see this think of  $\mathcal{Q} \times W$  as an infinite book: each leaf is  $\mathcal{Q} \times I$  and the spine is  $\mathcal{Q} \times \{q\}$ . Pick a decreasing sequence of irrationals converging (in  $\mathbf{R}$ ) to 0, say  $\alpha_1, \alpha_2, \dots$ ; in the  $n$ th leaf of the book pick a sequence  $\{x_{n,i}\}_{i=1}^\infty$  in  $\mathcal{Q} \times (0, 1]$  which converges to  $(\alpha_n, 0)$  in  $\mathbf{R} \times I$ . Then  $F = \bigcup_n \{x_{n,i}\}_{i=1}^\infty$  is sequentially closed, but not closed since  $(p, q)$  is in its closure.

If  $X$  is any topological space, one may impose a sequential topology on it by taking as open sets of the new topology all the sequentially open sets of the original topology. We denote this new space by  $S(X)$  and call it the *sequential coreflection* of  $X$ . If  $X$  was sequential,  $S(X)$  will have the same topology as  $X$ ; if  $X$  was not sequential,  $S(X)$  will have a strictly finer topology. For a more extensive discussion, see [D]. We will need the fact that if  $X$  is a topological space and  $A$  is a closed subset of  $X$  which is sequential in its inherited topology, then the topology which  $A$  inherits from  $S(X)$  is the same one it inherits from  $X$ . To see this, observe that  $S(X)$  has the same convergent sequences as  $X$ ; the topology  $A$  inherits from  $X$  is

sequential by hypothesis and the topology it inherits from  $S(X)$  is sequential since  $A$  is closed in  $S(X)$ . Thus both topologies are completely determined by their convergent sequences which are the same.

**2. Free topological groups and  $k_\omega$ -spaces.** Let  $(X, p)$  be a Tychonoff space with basepoint  $p$ . The *Graev free topological group* over  $(X, p)$  is a topological group  $F_G(X, p)$  which is algebraically the free group on  $X \setminus \{p\}$  and whose topology is the finest topology compatible with the group structure making the “insertion of generators”  $\eta: (X, p) \rightarrow F_G(X, p)$  continuous ( $\eta(p)$  is the group identity  $e$ ).  $F_G(X, p)$  has the usual properties associated with the word “free”; in particular, any continuous pointed map  $f: (X, p) \rightarrow (G, e)$  into a topological group extends uniquely to a continuous homomorphism  $\hat{f}: F_G(X, p) \rightarrow (G, e)$ .

While the topology of  $F_G(X, p)$  can be unpleasant in general [F-O-T], [H-M], it is tractable if  $X$  is a  $k_\omega$ -space. A topological space is called a  $k_\omega$ -space when it is the weak union of an increasing sequence of compact Hausdorff subspaces. The  $k_\omega$ -spaces are very well behaved [S], [Fr-T], [O<sub>1</sub>]: they are closed hereditary, finitely productive, preserved by countable disjoint (or pointed) unions, and preserved by Hausdorff quotients.

The spaces  $S_n$  and  $S_\omega$  are  $k_\omega$ -spaces.

If  $(X, p)$  is a  $k_\omega$ -space, then  $F_G(X, p)$  is topologically (and algebraically) the weak union of the subspaces  $(F_G(X, p))_n$  consisting of words of reduced length  $\leq n$ . Further, each of the subspaces  $(F_G(X, p))_n$  is the quotient of the product  $(X \cup_p X)^n$  (the cartesian product of  $n$  factors, each a pointed union of two copies of  $X$ ) in a natural way. For details, see [O<sub>1</sub>].

If  $G$  is a topological group and  $A$  is a subset of  $G$ , we say that  $A$  *generates*  $G$  provided that  $A$  algebraically generates  $G$  and that  $G$  has the finest topology compatible with both the group structure and the original topology on  $A$  (that is, any strictly finer group topology on  $G$  would induce a strictly finer topology on the subset  $A$ ). Theorem 1 of [MMO] states that if  $A$  generates  $G$  and  $A$  is a  $k_\omega$ -space, then  $G$  is also a  $k_\omega$ -space in a natural way. We will need the following related result:

**LEMMA 2.1.** *Let  $G$  be a topological group and  $A$  a subset which generates it and contains the identity. Suppose  $A$  is a  $k_\omega$ -space. Then the map  $\hat{f}: F_G(A, e) \rightarrow (G, e)$  which extends the inclusion  $f: A \hookrightarrow G$  is a quotient map.*

**PROOF.** Write  $A = \cup_i A_i$ , where the basepoint is in  $A_1$  and the  $A_i$ 's form an increasing sequence of compact Hausdorff subspaces of  $A$  which determines the topology of  $A$ . Then by the formulation in [MMO], the  $i$ th compact Hausdorff subset of  $F_G(A, e)$  (in a sequence determining the topology) may be taken to be the set of words which are products of at most  $i$  elements of  $\eta(A_i)$ , or their inverses, and the  $i$ th compact subset of  $G$ , in the quotient topology under  $\hat{f}$ , may be taken to be the set of products of at most  $i$  elements of  $\hat{f}\eta(A_i)$ , or their inverses. However, this is precisely the topology that  $G$  already has as a group generated by  $A$ , according to the proof of Theorem 1 of [MMO].

### 3. Sequential conditions in free groups.

**THEOREM 3.1.** *A  $k_\omega$ -space  $(X, p)$  is sequential if and only if its Graev free topological group,  $F_G(X, p)$ , is sequential.*

**PROOF.** Since  $(X, p)$  can be embedded as a closed subspace of  $F_G(X, p)$ , it must be sequential if  $F_G(X, p)$  is. On the other hand, suppose  $(X, p)$  is sequential; write  $X = \bigcup_i X_i$ , where  $p$  is in  $X_1$  and the  $X_i$ 's form an increasing sequence of compact Hausdorff subspaces of  $X$  which determines the topology of  $X$ . Then each of the spaces  $X_i \cup_p X_i$  is sequential and compact, so (by the theorem of Boehme) each  $(X_i \cup_p X_i)^2$ , and by induction each  $(X_i \cup_p X_i)^n$ , is sequential. Thus the weak union (in fact,  $k_\omega$ -decomposition)  $(X \cup_p X)^n = \bigcup_i (X_i \cup_p X_i)^n$  is sequential for each  $n$ . It follows that the quotients  $(F_G(X, p))_n$  are sequential, and so the weak union  $F_G(X, p)$  must be also.

**COROLLARY 3.2.** *Let  $G$  be a topological group generated (in our special sense) by a subset  $A$ . If  $A$  is a sequential  $k_\omega$ -space, then so is  $G$ .*

**PROOF.**  $G$  is a  $k_\omega$ -space by [MMO]. It is sequential since by Lemma 2.1, it is a quotient of the sequential space  $F_G(A, p)$ .

The free product  $G * H$  of two topological groups  $[Gr_2]$ ,  $[O_2]$  is generated by  $G \cup_e H$ . Hence:

**COROLLARY 3.3.** *Let  $G$  and  $H$  be topological groups which are sequential  $k_\omega$ -spaces. Then their free product  $G * H$  is also a sequential  $k_\omega$ -space.*

From Theorem 3.1 we see that there are free topological groups which are sequential. Naturally, one wonders what happens when the " $k_\omega$ " requirement is dropped. As before, if the Graev free topological group  $F_G(X, p)$  is sequential, then  $(X, p)$  must be sequential. The following example shows that the implication in the other direction fails:

**EXAMPLE 3.4.** Let  $X$  be the pointed union of  $(\mathcal{Q}, p)$  with  $(W, q)$ . Since  $\mathcal{Q}$  and  $W$  are sequential, so is  $X$ . But by [F-O-T],  $F_G(X, p)$  contains a closed subspace homeomorphic to  $\mathcal{Q} \times W$ ; since  $\mathcal{Q} \times W$  is not sequential, neither is  $F_G(X, p)$ .

In Theorem 6.6 of [D], Dudley proves the following: let  $P$  denote the set of all real functions of a real variable, with the topology of pointwise convergence. Let  $S(P)$  denote the sequential coreflection of  $P$ .  $P$  is of course a topological group; however, assuming the continuum hypothesis,  $S(P)$  is not a topological group: addition of functions is discontinuous. If we use the space  $F_G(X, p)$  of the above example, we do not need the continuum hypothesis to obtain an example of a topological group whose sequential coreflection is not a compatible topology:

**EXAMPLE 3.5.** Let  $X$  be as in Example 3.4.  $F_G(X, p)$  is a topological group. Since it is not sequential, its sequential coreflection  $S(F_G(X, p))$  has a strictly finer topology, which, however, induces the original topology on the closed sequential subset  $\eta(X)$ . However, the free topology on  $F_G(X, p)$  is the finest such topology compatible with the group operation; hence, the group operation is discontinuous in the topology of  $S(F_G(X, p))$ .

We now turn to the problem of determining the sequential order of a free topological group.

**LEMMA 3.6.** *Let  $(X, p)$  be a Tychonoff space and suppose there is a sequence of distinct terms  $x_1, x_2, x_3, \dots$  in  $X$  converging to  $p$ . Let  $T$  be the set  $\{p, x_1, x_2, \dots\}$ . Then  $F_G(T, p)$  is contained in  $F_G(X, p)$  as a closed subgroup.*

**PROOF.** For a similar result, see Proposition 5.3 of [O<sub>1</sub>]. We give an argument using a method of proof developed in [H-M-T]. Let  $\beta X$  be the Stone-Ćech compactification of  $X$ . The inclusion  $T \subset X \subset \beta X$  is an inclusion of the compact set  $T$  in the compact space  $\beta X$  and yields continuous homomorphisms  $F_G(T, p) \rightarrow F_G(X, p) \rightarrow F_G(\beta X, p)$  where it may be easily checked (since  $F_G(\beta X, p)$  is a  $k_\omega$ -space) that  $F_G(T, p) \rightarrow F_G(\beta X, p)$  is a closed embedding. Hence the image of  $F_G(T, p)$  is closed in  $F_G(X, p)$ .

**THEOREM 3.7.** *Let  $(X, p)$  be a Tychonoff space and suppose there is a sequence of distinct terms  $x_1, x_2, x_3, \dots$  in  $X$  converging to  $p$ . Then there is a closed embedding  $f: (S_\omega, s_0) \rightarrow F_G(X, p)$ .*

**PROOF.** In view of Lemma 3.6, it will suffice to produce a closed embedding  $f: (S_\omega, s_0) \rightarrow F_G(T, p)$ . Enumerate the sequences of  $S_\omega$  as follows: denote the single sequence of  $S_1$  by  $t_1 = s_1, s_2, s_3, \dots$ . Denote the sequence of  $S_2$  converging to  $s_1$  by  $t_2 = s_{1,1}, s_{1,2}, s_{1,3}, \dots$ . Use a diagonalization process to enumerate all the sequences of  $S_\omega$ ; the limits of the sequences  $t_3, t_4, t_5, t_6, t_7, t_8, \dots$  are respectively  $s_{1,1}, s_2, s_{1,1,1}, s_{1,2}, s_{2,1}s_3, \dots$ . The basic idea in constructing the function  $f$  is to map each sequence  $t_i$  into the set of words of  $F_G(T, p)$  which have reduced length precisely  $i$ . Let  $f(s_0) = p$ . Let  $f(t_1) = T \setminus \{p\}$ , with  $f(s_n) = x_n$ . Let  $f(t_2) = x_1x_1, x_1x_2, x_1x_3, \dots$ ; let  $f(t_3) = x_1x_1x_1, x_1x_1x_2, x_1x_1x_3, \dots$ ; and let

$$f(t_4) = x_2x_1x_1x_1, x_2x_2x_2x_2, x_2x_3x_3x_3, \dots$$

Inductively, if the sequence  $t_i$  converges to  $s_{j,k,\dots,m}$ , then  $f(t_i) = f(s_{j,k,\dots,m})x_1^r, f(s_{j,k,\dots,m})x_2^r, \dots$ , where  $r = i - (j + k + \dots + m)$ . Note that in view of the way the  $s_{j,k,\dots}$  were enumerated and the way the  $i$  are defined, the exponents  $r$  are always positive, and the function  $f$  is clearly one-to-one. Since  $f$  was chosen to preserve sequential convergence and  $S_\omega$  is sequential,  $f$  is continuous. That  $f$  is a closed embedding follows readily from the fact that the intersection of its image with each  $(F_G(T, p))_n$  consists of precisely  $n$  convergent sequences with their limits.

**COROLLARY 3.8.** *Let  $(X, p)$  be a Tychonoff space and suppose it contains some point which is the limit of a nonconstant sequence. Then  $F_G(X, p)$  contains a closed subspace homeomorphic to  $(S_\omega, s_0)$ .*

**PROOF.** A convergent subsequence of distinct terms may be extracted from the given sequence in  $X$ . If the limit point is  $p$ , Theorem 3.7 applies. But by [Gr<sub>1</sub>],  $F_G(X, p)$  is (up to isomorphism of topological groups) independent of the choice of basepoint  $p$  in  $X$ .

Since  $(S_\omega, s_0)$  is sequential of order  $\omega_1$ , we obtain:

**THEOREM 3.9.** *Let  $(X, p)$  be any Tychonoff space, and suppose  $F_G(X, p)$  is sequential but not discrete. Then it is sequential of order  $\omega_1$ .*

**PROOF.** As was noted before Example 3.4,  $X$  is sequential, and  $X$  is not discrete (or  $F_G(X, p)$  would be also). Hence  $X$  contains a nonconstant convergent sequence, and  $F_G(X, p)$  contains a copy of  $(S_\omega, s_0)$ . Hence the sequential order of  $F_G(X, p)$  is at least  $\omega_1$ ; since that is the maximum possible sequential order of any space, it must be exactly  $\omega_1$ .

It would be nice to take Corollary 3.8 one step further, leading to the following:

**Question 3.10.** Let  $F_G(X, p)$  contain some nontrivial convergent sequence. Must it contain a copy of  $(S_\omega, s_0)$ ? A closed copy?

An affirmative answer would follow from an affirmative answer to:

**Question 3.11.** Let  $F_G(X, p)$  contain a nontrivial convergent sequence. Need  $(X, p)$  contain a nontrivial convergent sequence?

**4. Nongroupability of  $S_\omega$ .** Since we have now embedded  $(S_\omega, s_0)$  as a closed subspace of  $F_G(X, p)$  for many spaces  $(X, p)$ , it is natural to ask if it can be embedded as a subgroup. Clearly, the embedding given here does not make it a subgroup, since it includes all elements of  $\eta(T) \subset F_G(T, p)$  but not all products of three such elements. In fact, no embedding can make it a subgroup. The following result is even stronger:

**THEOREM 4.1.** *There is no topological group homeomorphic to  $(S_\omega, s_0)$ .*

Loosely, our strategy will be to show that while diagonals generally fail to converge in  $S_\omega$ , in a topological group diagonals generally do converge. We need this lemma:

**LEMMA 4.2.** *In  $S_\omega$  let  $y_0$  be a fixed but arbitrary point. Let  $y_1, y_2, y_3, \dots$  be a sequence converging to  $y_0$ . For each  $i$  let  $y_{i,1}, y_{i,2}, y_{i,3}, \dots$  be a sequence converging to  $y_i$ . Suppose all the points  $y_0, y_i, y_{i,j}$  are distinct. Then there is a function  $f: \mathbf{N} \rightarrow \mathbf{N}$  ( $\mathbf{N}$  is the natural numbers) with  $f(i) \geq i$  for all  $i$ , such that the sequence  $y_{i,f(i)}$ ,  $i = 1, 2, 3, \dots$  fails to converge.*

**PROOF.** We must describe the topology of  $S_\omega$  in more detail. Consider the labelling of sequences in  $S_\omega$  given in the proof of Theorem 3.7; let  $T_n$  denote the sequence  $t_n$  and its limit. Instead of considering  $S_\omega$  to be the weak union of the spaces  $S_n$ , we may regard it as a quotient of the disjoint union of the sequences  $T_n$ . With this viewpoint, we see that a subset  $A$  of  $S_\omega$  is closed if and only if every intersection  $A \cap T_n$  is closed. If  $n \neq m$ ,  $T_n$  and  $T_m$  intersect in at most one point, and if there is such a point it is the (unique) limit point of exactly one of  $T_n$  or  $T_m$ . Each point of  $S_\omega$  is the limit point of exactly one  $T_n$  and, except for  $s_0$ , a nonlimit point of exactly one other. A sequence converges only if it is eventually in some  $T_n$ , and then, if it is not eventually constant, it converges to the limit point of that  $T_n$ .

Now,  $y_0$  is the limit point of some  $T_n$ , say  $T_{n_0}$ , and the sequence  $y_1, y_2, y_3, \dots$  is eventually in  $T_{n_0}$ . Each  $y_i$  is the limit point of some  $T_n$ , say  $T_{n_i}$ , and the sequence  $y_{i,1}, y_{i,2}, y_{i,3}, \dots$  is eventually in  $T_{n_i}$ . Pick  $f$  so that  $f(i) \geq i$  and  $y_{i,f(i)} \in T_{n_i}$  for each  $i$ .

Now  $y_{i,f(i)}$  is a nonlimit point of  $T_n$  for each  $i$ ; hence at most two  $y_{i,f(i)}$  lie in any  $T_n$ , the sequence  $y_{i,f(i)}$ ,  $i = 1, 2, 3, \dots$ , is not eventually in any  $T_n$ , and so this sequence cannot converge.

PROOF OF THEOREM 4.1. Suppose  $S_\omega$  to be a topological group. Let  $x_0$  be a point in  $S_\omega$  and let  $x_1, x_2, x_3, \dots$  be a sequence converging to  $x_0$  such that all the points  $x_0, x_1, x_2, \dots$  are distinct and none is the group identity. Denote the multiplication on  $S_\omega$  by  $m(\ , \ )$ .

We shall select some points in  $S_\omega$  to fill the roles of the points in the statement of Lemma 4.2. Let  $y_0$  be  $m(x_0, x_0)$ . Let  $y_i$  be  $m(x_0, x_i)$  for  $i = 1, 2, 3, \dots$ . Then  $y_0$  and the  $y_i$  are distinct and  $\lim y_i = y_0$ . We next choose the  $y_{1,j}$ ,  $j = 1, 2, 3, \dots$  as follows: the sequence  $m(x_j, x_1)$ ,  $j = 1, 2, 3, \dots$ , converges to  $m(x_0, x_1)$ . It is eventually disjoint from the sequence  $m(x_0, x_i)$  because the two sequences have different limits. Hence there is a positive integer  $k_1$  such that for  $j \geq k_1 + 1$ , the points  $m(x_j, x_1)$  are distinct from all points chosen thus far. Pick as  $y_{1,j}$  the point  $m(x_{k_1+j}, x_1)$ . Having picked sequences  $y_{i,j}$ ,  $j = 1, 2, 3, \dots$ , for  $i < n$ , proceed by induction: Note that  $m(x_j, x_n)$ ,  $j = 1, 2, 3, \dots$ , converges to  $m(x_0, x_n)$  and is eventually disjoint (for  $j > k_n$ ) from the set consisting of  $y_0$ , the  $y_i$ 's, and all the previously chosen  $y_{i,j}$ 's; pick  $k_n$  sufficiently large and let  $y_{n,j} = m(x_{k_n+j}, x_n)$ .

The  $y_0$ ,  $y_i$ 's, and  $y_{i,j}$ 's we have selected from  $S_\omega$  meet the conditions of Lemma 4.2. Now let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be any function with  $f(i) \geq i$  for all  $i$ .  $\lim x_i = x_0$  (all limits are taken as  $i \rightarrow \infty$ ); also  $\lim x_{f(i)+k_i} = x_0$  since if  $i \geq M$ ,  $f(i) + k_i \geq M$ . Hence in  $S_\omega \times S_\omega$ ,  $\lim(x_{f(i)+k_i}, x_i) = (x_0, x_0)$ . Now by continuity of multiplication in  $S_\omega$ ,

$$\lim y_{i,f(i)} = \lim m(x_{f(i)+k_i}, x_i) = m(x_0, x_0) = y_0$$

contradicting Lemma 4.2 and completing the proof of Theorem 4.1.

### REFERENCES

- [A-Fr] A. V. Arhangel'skii and S. P. Franklin, *Ordinal invariants for topological spaces*, Michigan Math. J. **15** (1968), 313-320.
- [B] T. K. Boehme, *Linear s-spaces*, (Proc. Sympos. Convergence Structures, Univ. of Oklahoma, Norman, 1965), Univ. of Oklahoma, Norman, 1965.
- [D] R. M. Dudley, *On sequential convergence*, Trans. Amer. Math. Soc. **112** (1964), 483-507; *Corrections*, ibid. **148** (1970), 623-624.
- [F-O-T] T. H. Fay, E. T. Ordman and B. V. S. Thomas, *The free topological group over the rationals*, General Topology and Appl. **10** (1979), 33-47.
- [Fr<sub>1</sub>] S. P. Franklin, *Spaces in which sequences suffice*. I, Fund. Math. **57** (1965), 107-115.
- [Fr<sub>2</sub>] \_\_\_\_\_, *Spaces in which sequences suffice*. II, Fund. Math. **61** (1967), 51-56.
- [Fr-T] S. P. Franklin and B. V. S. Thomas, *A survey of  $k_\omega$ -spaces*, (Topology Proceedings, vol. 2, no. 1, Proc. 1977 Topology Conf., Baton Rouge, La.), Auburn Univ., Auburn, Ala., 1978, pp. 111-124.
- [Gr<sub>1</sub>] M. I. Graev, *Free topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945), 3-64; English transl., Amer. Math. Soc. Transl. (1) **8** (1962), 305-364.
- [Gr<sub>2</sub>] \_\_\_\_\_, *On free products of topological groups*, Izv. Akad. Nauk SSSR Ser. Mat. **14** (1950), 343-354. (Russian)
- [H-M] David C. Hunt and Sidney A. Morris, *Free subgroups of free topological groups*, (Proc. Second Internat. Conf. Theory of Groups, Canberra, 1973), Lecture Notes in Math., vol. 372, Springer-Verlag, Berlin and New York, 1974, pp. 377-387.
- [H-M-T] J. P. L. Hardy, Sidney A. Morris and H. B. Thompson, *Applications of the Stone-Ćech compactification to free topological groups*, Proc. Amer. Math. Soc. **55** (1976), 160-164.
- [M-M-O] John Mack, Sidney A. Morris and Edward T. Ordman, *Free topological groups and the projective dimension of a local compact abelian group*, Proc. Amer. Math. Soc. **40** (1973), 303-308.



[O<sub>1</sub>] Edward T. Ordman, *Free  $k$ -groups and free topological groups*, *General Topology and Appl.* **5** (1975), 205–219.

[O<sub>2</sub>] ———, *Free products of topological groups which are  $k_\omega$ -spaces*, *Trans. Amer. Math. Soc.* **191** (1974), 61–73.

[R] M. Rajagopalan, *Sequential order and spaces  $S_n$* , *Proc. Amer. Math. Soc.* **54** (1976), 433–438.

[S] N. E. Steenrod, *A convenient category of topological spaces*, *Michigan Math. J.* **14** (1967), 133–152.

DEPARTMENT OF MATHEMATICS, NEW ENGLAND COLLEGE, HENNIKER, NEW HAMPSHIRE 03242 (Current address of E. T. Ordman)

DEPARTMENT OF MATHEMATICAL SCIENCES, MEMPHIS STATE UNIVERSITY, MEMPHIS, TENNESSEE 38152

*Current address* (B. V. Smith-Thomas): Department of Mathematics, University of Alabama, Birmingham, Alabama 35294