Simple Closed Curves and Two Consecutive Steiner Symmetrizations

E 2856 [1980, 755]. *Proposed by F. W. Luttmann, Sonoma State University.*

Does there exist a simple closed curve, other than the circle, such that two consecutive Steiner symmetrizations with respect to two orthogonal lines always produce a circle?

Solution by Peter Loomis, Lockheed Corporation, Sunnyvale, CA. The answer is no. Suppose K is a compact convex set in the plane with the property that every pair of consecutive Steiner symmetrizations in orthogonal lines turns it into a circle. We show that K is a circle. Let some direction be chosen, and let the width of K in that direction be w. If K is symmetrized with respect to a line in this direction, its width in this direction is unchanged. If this new body is now symmetrized in the orthogonal direction, the width in the original direction is again unchanged because there is a section of length w in this direction. But a circle is obtained by hypothesis, and the width of this circle is its diameter. The width w is therefore equal to the diameter of this circle. Since the circles obtained by each pair of orthogonal symmetrizations are all of the same area as K and hence of the same diameter, K's width in every direction is the same, namely equal to the diameter of the circle of equal area. So K is a curve of constant width. It is well known that such a curve has as its perimeter that of the circle of the same width. But if K has both the perimeter and the area of a circle, it must *be* a circle by the isoperimetric inequality.

An Analytic Characterization of Egyptian Fractions

E 2934 [1982, 212]. *Proposed by R. P. Boas, Northwestern University.*

Let f be a real-valued continuous function on $[0,1]$ and let h be a number between 0 and 1. Suppose the average of f over each subinterval of $(0,1)$ of length h is less than 1. Can the average of f over $[0,1]$ be greater than 1?

Solution by Edward T. Ordman, Memphis State University. This can happen if $h \neq 1/n$ for any positive integer n; for small enough ε, greater than 0, $f(x) = 1 - \varepsilon + \sin(2\pi x/h)$ is an example. Note that $\int \sin(2\pi x/h)\,dx$ is 0 over any interval of length exactly h and positive over the interval $[0,1]$ since 1 is not a multiple of h. Let δ be the value of the integral over $[0,1]$ and pick a positive ε less than δ. Then $f(x)$ has average value $1 - \varepsilon$ over any subinterval of length h and average value $1 - \varepsilon + \delta > 1$ over $[0,1]$.

If $h = 1/n$ for some positive integer n, then the average value of f cannot exceed 1. To see this, cover $[0,1]$ "approximately" with a collection of subintervals $[\alpha, h+\alpha]$, $[h, 2h], \ldots,$ $[(n-2)h, (n-1)h]$, $[1-h-\alpha, 1-\alpha]$, where $nh = 1$. The integral of f over each of these subintervals is less than h, so the integral of f over $[\alpha, 1-\alpha]$ is less than $nh = 1$. Since f is continuous on $[0,1]$, we may take the limit as α approaches 0 and find that the integral of f over $[0,1]$ does not exceed 1.

Also solved by 57 other readers and the proposer. Several solutions were incomplete.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by February 28, 1985. The solver's full post-office address should be on each sheet.

6469. *Proposed by A. Wilansky, Lehigh University.*

1. Suppose that a countable set E of real sequences has the property that $E^{\beta} = L = \{x\colon \Sigma|x_n| < \infty\}$. Show that E has a finite subset with the same property. (Notation: $E^{\beta} = \{x\colon \Sigma x_i y_i \text{ converges } \forall y \in E\}$.)
2. Does such a set E exist?