ALMOST LOCALLY INVARIANT TOPOLOGICAL GROUPS

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ABSTRACT

Maximally almost periodic groups and locally invariant groups have been studied extensively in the literature. Maximally almost periodic groups are those admitting continuous monomorphisms into compact groups; locally invariant groups are those in which every neighbourhood of the identity contains a neighbourhood invariant under inner automorphisms. In this paper a study is made of almost locally invariant groups, which are groups admitting continuous monomorphisms into locally invariant groups. This class includes all maximally almost periodic groups and all locally invariant groups, but there exist locally compact almost locally invariant groups which are neither locally invariant nor maximally almost periodic. A locally compact almost locally invariant group which is connected or locally connected is locally invariant. The class of almost locally invariant groups is closed under passage to subgroups, direct products, and free products, but not quotients.

1. *Introduction*

Locally invariant (SIN) groups and maximally almost periodic groups have been studied for some time in the literature; (see **[2]** and **[3]**.) This paper is devoted to the study of almost locally invariant groups; that is, those groups which admit continuous monomorphisms into locally invariant Hausdorff groups. This class of groups includes the classes of locally invariant groups and maximally almost periodic groups and is shown to coincide with them for the case of connected or locally connected locally compact groups. It is noteworthy that the free product **[1, 9]** of almost locally invariant groups is almost locally invariant, while a free product of topological groups is locally invariant only in trivial cases **[10]**.

2. *Preliminaries*

Throughout this paper, unless otherwise stated, the word *group* will mean Hausdorff topological group. A group will be called *locally invariant* if every neighbourhood of the identity e contains an invariant neighbourhood of e; that is, a neighbourhood invariant under the inner automorphisms of the group. A group will be called *almost locally invariant* if it admits a continuous monomorphism into a locally invariant group. Clearly every locally invariant group is almost locally invariant. We note that all compact groups, discrete groups, and Abelian groups are locally invariant (and thus almost locally invariant); on the other hand, the real plane with the usual topology and operation $(a, b)(c, d) = (a+c, b+e^a d)$ is connected and locally compact without being almost locally invariant. A group is called *maximally almost periodic* (MAP) if it admits a continuous monomorphism into a compact group. Since every compact group is locally invariant, every MAP group is almost locally invariant.

It is well known **[2]** that in general a group may be either locally invariant or MAP without having the other property, but that the two properties are equivalent for connected, locally compact groups, or, more generally, for locally compact groups G such that G/G_0 is compact, where G_0 is the component of the identity. Our principal result is

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THEOREM A. *Let G be a locally compact group for which G/G_0 is compact, where G_0 is the component of the identity. Then the following conditions are equivalent*:

(a) *G is almost locally invariant*;

(b) *G is locally invariant*;

(c) *G is* MAP.

The proof of Theorem A is in §4. §3 is devoted to more routine properties of almost locally invariant groups.

3. *Properties of almost locally invariant groups*

The following property will be useful later. We show the implication that we shall need, leaving open the question whether the converse holds.

PROPOSITION 1. *Let G be an almost locally invariant group. Then there cannot be nets $\{x_\delta\}$ and $\{g_\delta\}$ (for $\delta \in \Delta$ the same directed set) in G such that $\{x_\delta\}$ converges to the identity e and $\{g_\delta^{-1} x_\delta g_\delta\}$ converges to a point $y \neq e$.*

Proof. Recall that all groups are assumed Hausdorff and thus nets have at most one limit point. Suppose that G is almost locally invariant and such nets $\{x_\delta\}$ and $\{g_\delta\}$ exist. Let $f : G \to H$ be a continuous monomorphism into a locally invariant group H. Now, by continuity, $\{f(g_\delta^{-1} x_\delta g_\delta)\}$ converges to $f(y) \neq e$; but, since H is locally invariant, the convergence of $\{f(x_\delta)\}$ to e implies the convergence of $\{f(g_\delta)^{-1} f(x_\delta) f(g_\delta)\}$ to e. As $\{f(g_\delta^{-1} x_\delta g_\delta)\}$ cannot converge to both e and $f(y) \neq e$, we obtain a contradiction.

Another invariant property which has received some attention is [IN]; a group has this property if there exists a compact invariant neighbourhood of the identity. A locally compact locally invariant group has property [IN] but [IN] does not in general imply local invariance [2]. [IN] together with almost local invariance, however, implies local invariance.

THEOREM 1. *If G is an almost locally invariant group and has a compact invariant neighbourhood K of e, then G is locally invariant.*

Proof. Suppose G is not locally invariant and let N be a neighbourhood of e containing no invariant neighbourhood of e. Let $\mathscr{F}$ denote the filter of neighbourhoods of e contained in K, so that if $F \in \mathscr{F}$, there is a $g_F \in G$ with

$$g_F^{-1} F g_F \cap (K - N) \neq \varnothing$$

(this is possible since $\bigcup_{g \in G} g^{-1} Fg$ is an invariant neighbourhood of e contained in K and cannot be contained in N). Choose $x_F \in F$ such that $g_F^{-1} x_F g_F \in K - N$. Now by the compactness of K the net $\{g_F^{-1} x_F g_F\}$ has a subnet $\{g_\delta^{-1} x_\delta g_\delta\}$ converging to some $y \in K$; since this subnet remains outside N, $y \neq e$. But the corresponding subnet $\{x_\delta\}$ converges to e, since it is eventually in each $F \in \mathscr{F}$. Thus, by Proposition 1, G is not almost locally invariant—a contradiction.

We recall that the class of locally invariant groups is closed under the formation of subgroups, finite or infinite direct products and quotients. The class of almost locally invariant groups is clearly closed under the formation of subgroups and finite or infinite direct products. It is also closed under passing to a finer group topology on a given group. It is not, however, closed under the formation of quotients. In fact every group is a quotient group of an almost locally invariant group. This is most easily seen by noting that every topological group G is a quotient of its free topological group $F(G)$ and that every free topological group is MAP and hence almost locally invariant. (For details on free topological groups, see **[4]** and **[6]**.)

The following partial result is then of interest and will be useful in proving Theorem A.

PROPOSITION 2. *If G is an almost locally invariant group and K is a compact normal subgroup, then G/K is almost locally invariant.*

Proof. Let $f : G \to H$ be a continuous monomorphism of G onto a locally invariant group H. Consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \phi \big\downarrow & & \big\downarrow \gamma \\ G/K & \xrightarrow[g]{} & H/f(K) \end{array}$$

where ϕ and γ are the natural quotient maps and g is the induced map. Since K is compact, $f(K)$ is closed in H and hence $H/f(K)$ is indeed Hausdorff. Clearly g is a continuous monomorphism and $H/f(K)$ is locally invariant. Thus G/K is almost locally invariant.

The class of almost locally invariant groups is closed under another operation under which the class of locally invariant groups is not; the operation of free product. The *free product* of a class of topological groups was introduced by Graev **[1]**. It has as underlying abstract group the free product of the abstract groups of the factors; if the factors are Hausdorff groups so is the free product. It has been shown **[5** and **9]** that the free product of MAP groups is again MAP; on the other hand, under quite general conditions **[10]** the free product of locally invariant groups fails to be locally invariant.

THEOREM 2. *A (finite or infinite) free product of groups is almost locally invariant if and only if each factor is almost locally invariant.*

Proof. As each factor is (isomorphic to) a subgroup of the product, almost local invariance of the product implies that of each factor. To show the converse we recall briefly the proof of the main theorem of **[12]**. The goal there was to prove (independently of Graev) that a free product of locally invariant groups is Hausdorff. The proof uses the fact that each factor, being locally invariant, has enough two-sided invariant pseudometrics to distinguish points, and uses these pseudometrics to build enough two-sided invariant pseudometrics on the free products to distinguish points there. These pseudometrics yield a locally invariant Hausdorff topology on the free product which is, however, coarser in general than the free product topology.

Now suppose that for each $\alpha \in A$, (G_α, τ_α) is an almost locally invariant group; then each G_α admits a coarser topology τ_α' so that (G_α, τ_α') is a locally invariant group. The discussion summarized above shows that there is a topology s on the abstract free product $\Pi^*(G_\alpha : \alpha \in A)$ making it into a locally invariant (Hausdorff) group and inducing the topology τ_α' on each factor G_α.

Now there is a continuous injection $(G_\alpha, \tau_\alpha) \to (\Pi^*(G_\alpha : \alpha \in A), s)$ for each α and by the universal property of free products there is a unique continuous homomorphism of the free product $\Pi^*((G_\alpha, \tau_\alpha) : \alpha \in A)$ into $\Pi^*((G_\alpha : \alpha \in A), s)$ which from algebraic considerations is the algebraic identity map. Hence we have a continuous monomorphism of $\Pi^*((G_\alpha, \tau_\alpha) : \alpha \in A)$ into a locally invariant (Hausdorff) group, as required.

In fact it follows that the proof in **[12]** shows Hausdorffness for free products of almost locally invariant groups as well as locally invariant groups.

We conclude this section by noting that **[2]** gives reference to examples of locally compact groups which are locally invariant but not MAP and which are MAP but not locally invariant. This gives us examples of locally compact, almost locally invariant groups which are either not MAP or not locally invariant. To find a locally compact locally invariant group which is neither MAP nor locally invariant, simply take the direct product of two groups one of which is locally invariant, locally compact but not MAP and the other of which is locally compact MAP but non-locally invariant. Non-locally compact examples of this type can be manufactured using free products **[10, 11]**.

4. *Proof of Theorem A*

We shall require the following well-known theorem **[8**; p. 175].

THEOREM 3. (a) *Let G be any locally compact group such that G/G_0 is compact, where G_0 is the component of the identity. Let V be any neighbourhood of e. Then V contains a compact normal subgroup H of G such that G/H is isomorphic to a Lie group.*

(b) *If P is the direct product of all such G/H for a fixed G, then G is isomorphic to a subgroup of P.*

Proof. (a) appears in **[8**; p. 175]. (b) follows routinely, where $f : G \to P$ is simply the quotient map $G \to G/H$ in the co-ordinate corresponding to G/H; see **[7**; p. 116].

The only Lie group properties we require are local compactness and local connectedness. We first show that almost local invariance implies local invariance in the presence of these two properties.

THEOREM 4. *If a group G is locally compact, locally connected, and almost locally invariant, then it is locally invariant.*

Proof. Suppose G is not locally invariant and let K be an open neighbourhood of e such that the closure $\overline{K}$ of K is compact and contains no invariant neighbourhood. Let $\{N\}$ be a neighbourhood basis at e consisting of connected sets. Thus for each such N there is a $g_N \in G$ such that $g_N{}^{-1} N g_N - \bar{K} \neq \varnothing$ (otherwise $\bigcup g^{-1} Ng$, $g \in G$ would be an invariant subneighbourhood of $\bar{K}$). Now $g_N{}^{-1} N g_N$ meets K (at e) and meets $G - \bar{K}$; hence it meets $\bar{K} - K$, for otherwise its intersections with K and $G - \bar{K}$ would partition it into disjoint, non-empty, relatively open sets, contradicting its connectedness. Choose $x_N \in N$ such that $g_N{}^{-1} x_N g_N \in \bar{K} - K$. Since $\bar{K} - K$ is compact, an appropriate subnet $\{g_\delta{}^{-1} x_\delta g_\delta\}$ must converge to some $g \in \bar{K} - K$; and by Proposition 1, G is not almost locally invariant—a contradiction.

We are now able to complete the proof of Theorem A. The equivalence of (b) and (c) is well known **[2]** and we have observed that either implies (a). Suppose now that G is locally compact, G/G_0 is compact and G is almost locally invariant. Consider the groups G/H of Theorem 3. By Proposition 2, they are almost locally invariant. Since they are also locally compact and locally connected, Theorem 4 implies that they are locally invariant. Hence their product P is locally invariant and G, being isomorphic to a subgroup of P, is locally invariant, as desired.

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