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Blocking Sets for Paths of a Given Length

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Abstract

How many vertices must we delete from a graph in order no longer contains a path P_k on k vertices? We explore this question for various special graphs (hypercubes, square lattice graphs) as well as for general families.

1 Introduction

For basic definitions and notation, we refer the reader to standard texts on graph theory [3], [4], [7]. Given a graph G, let us say that $Z \subset V(G)$ is *k-blocking* if $G \setminus Z$ contains no path of order k. Given a graph G and an integer $k \geq 2$, we seek $\min |Z|$, where the minimum is taken over all k-blocking subsets $Z \subset V(G)$. The ratio of $\min |Z|$ to $|V(G)|$ will be called the *k-blocking ratio* of G.

The problem is suggested by various computer science applications. For the first, suppose that each vertex represents a state of a program (or finite state machine) and each edge a possible transition between states. It is desired to select a set of distinguished states (the set Z) so that the program will enter a distinguished state after at most k steps. (We assume, for this abstraction, that the program does not return to the same state during the k steps.)

For the second application, suppose that each vertex represents a computer and each edge a communication channel. We wish to record all "long distance" messages, specifically, all messages travelling at least k steps. If we could identify a subset Z of the computers representing all paths of order k, we could place recorders at only those processes.

Alon and Chung considered a problem of this type in connection with fault tolerant networks [1]. They obtained the following striking result.

Theorem (Alon, Chung). *For every $\epsilon > 0$ and every integer $k \geq 2$ there exists a graph G with (k/ϵ) vertices, maximum degree $\Delta = O(1/\epsilon^2)$, and k-blocking ratio at least $1 - \epsilon$.*

The proof of Alon and Chung uses the Ramanujan graphs studied by Lubotzky, Phillips, and Sarnak [8]. We shall study more commonplace examples (hypercubes, grid graphs) as well as general families of graphs.

Among the networks that have been used extensively in parallel computing are the hypercubes. Let Q_n denote the n-dimensional hypercube. This is the graph with vertex set $V(Q_n) = \{0,1\}^n$ in which $uv \in E(Q_n)$ if and only if the binary n-tuples u and v differ in exactly one component. Recursively, $Q_n = K_2 \times Q_{n-1}$, that is Q_n is obtained by taking two disjoint copies of Q_{n-1} and adding the $n-1$ edges that join corresponding vertices. [In general, $G_1 \times G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ in which $uv \in E(G_1 \times G_2)$ if $u_1v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2v_2 \in E(G_2)$.]

Proposition 1. *For a hypercube Q_n of dimension $n \geq 2$, the 2-blocking ratio is $1/2$, and so is the 3-blocking ratio.*

Proof. First, let us show that the 2-blocking ratio is no more than $1/2$. Let $w(v) = \sum_{i=1}^{n} v_i$ be the Hamming weight of vertex v. Note that Q_n is bipartite with bipartition $V(Q_n) = (W, Z)$ where $W = \{v|\ w(v) \text{ is even}\}$ and $Z = \{v|\ w(v) \text{ is odd}\}$. Then $|W| = |Z| = 2^{n-1}$ and Z (or W) is 2-blocking (and hence 3-blocking). In the other direction, first note that the recursive definition shows inductively that Q_n has a 2-factor consisting of 2^{n-2} C_4's. Any 3-blocking set must contain at least two vertices from each of these C_4's, and thus any 3-blocking set must contain at least 2^{n-1} vertices. □

2 Graphs of Large Degree

The following result uses the following well-known theorem of Erdős and Gallai [6]: a graph of order n that contains no P_k has at most $n(k-2)/2$ edges.

Theorem 1. *Suppose that for $i = 1, 2, 3, \ldots$ the graph G_i has order n_i and is regular of degree d_i, where $d_i \to \infty$ as $i \to \infty$. Given $k \geq 2$ and $\epsilon > 0$, there is an integer $N = N(\epsilon, k)$ such that for all $n_i > N$ the k-blocking ratio of G_i exceeds $\frac{1}{2} - \epsilon$.*

Proof. If G is a d-regular graph of order n, then G has $nd/2$ edges. Deleting any $(\frac{1}{2}-\epsilon)n$ vertices from G yields a graph with $(\frac{1}{2}+\epsilon)n$ vertices and at least $\frac{nd}{2}-(\frac{1}{2}-\epsilon)nd = nd\epsilon$ edges. If we assume that this graph contains no P_k, then the Erdős-Gallai theorem gives

$$d\epsilon < \left(\frac{1}{2}+\epsilon\right)\frac{k-2}{2},$$

which is clearly false provided d is sufficiently large. Since $d_i \to \infty$ for the given sequence (G_i), it follows that for all sufficiently large i the k-blocking ratio of G_i exceeds $\frac{1}{2}-\epsilon$. □

Corollary 1. *For any fixed $k \geq 2$ and any $\epsilon > 0$ the k-blocking ratio of Q_n is at least $(\frac{1}{2}-\epsilon)$ for all sufficiently large n.*

Proof. The hypercube Q_n is a regular graph of order 2^n and degree n. Hence the theorem applies. □

3 Graphs of Bounded Degree.

With W and Z disjoint subsets of $V(G)$, we shall denote by $E(W,Z)$ denote the edge set $\{wz \in E(G)|\ w \in W, z \in Z\}$.

Theorem 2. *Suppose G is a graph of order n and maximum degree Δ. (a) Some set of $\lfloor n\Delta/(\Delta+1)\rfloor$ vertices in G is 2-blocking. (b) Some set of $\lfloor n\Delta/(\Delta+2)\rfloor$ vertices in G is 3-blocking. Both results are sharp. Thus, for the family of all graphs with maximum degree Δ, the largest possible 2-blocking ratio is $\Delta/(\Delta+1)$ and the largest 3-blocking ratio is $\Delta/(\Delta+2)$.*

Proof. (a) Equivalently, $\alpha(G) \geq \lceil n/(\Delta+1)\rceil$. This follows immediately from Ramsey result of Chvátal [5] $r(T, K_m) = (m-1)q+1$ for any tree T with q edges (in particular, $T = K_{1,q}$). For another simple proof, note that if $W \subset V(G)$ is an independent set of order $\alpha(G)$ and $Z = V(G)\backslash W$, then Z is 2-blocking. Since W is a a maximal independent set, each vertex in Z is adjacent to at least one vertex in W. Hence

$$|Z| \leq |E(W,Z)| \leq (n-|Z|)\,\Delta,$$

which gives $|Z| \leq \lfloor n\Delta/(\Delta+1)\rfloor$. To see that this bound is sharp, consider the example $G \cong mK_{\Delta+1}$ where $m = n/(\Delta+1)$. In order to obtain an independent set, at least Δ vertices must be deleted from each component. Hence, any 2-blocking set contains at least $n\Delta/(\Delta+1)$ vertices.

(b) Let $W \subset V(G)$ be such that $P_3 \not\subset \langle W \rangle$, and, subject to this condition, $|W|$ is as large as possible. Further, assume that of all such sets with the maximum possible cardinality, W has been chosen so as to minimize the number of edges of $\langle W \rangle$. We claim that each vertex in $Z = V(G) \setminus W$ is adjacent to at least two vertices of W. Clearly, each $z \in Z$ is adjacent to at least one vertex in W. If $\Gamma(z) \cap W = \{w\}$ where w is isolated in $\langle W \rangle$, then $W' = W \cup \{z\}$ satisfies $|W'| > |W|$ and $P_3 \not\subset \langle W' \rangle$, a contradiction. Similarly, if $\Gamma(z) \cap W = \{w_1\}$ where $w_1 w_2$ is an isolated edge in $\langle W \rangle$, then $W' = (W \setminus \{w_1\}) \cup \{z\|$ satisfies $|W'| = |W|$ and $P_3 \not\subset \langle W' \rangle$. However, $\langle W' \rangle$ has fewer edges than $\langle W \rangle$, a contradiction. Hence

$$2|Z| \leq |E(W, Z)| \leq (n - |Z|)\,\Delta,$$

which gives $|Z| \leq n\Delta/(\Delta + 2)$. To see that this bound is sharp, suppose Δ is even, and consider $G \cong mCP(\Delta/2 + 1)$m where $m = n/(\Delta + 2)$ and $CP(r) = \overline{rK_2}$ denotes the *cocktail-party* graph [2, p. 17]. If fewer than $m\Delta$ vertices are deleted from G, then some component retains at least three vertices, and the subgraph spanned by these three contains P_3. Hence any 3-blocking set contains at least $m\Delta = n\Delta/(\Delta + 2)$ vertices. □

Next we prove that the examples used to show sharpness in the last theorem are unique. The following notation will be used: for $z \notin W$, write $\Gamma_W(z) = \Gamma(z) \cap W$.

Theorem 3. *(a) If G has order n, maximum degree Δ, and no set with fewer than $n\Delta/(\Delta + 1)$ vertices is 2-blocking, then $G \cong mK_{\Delta+1}$. (b) If G has order n, maximum degree Δ, and no set with fewer than $n\Delta/(\Delta + 2)$ vertices is 3-blocking, then Δ is even and $G \cong mCP(\Delta/2 + 1)$ where $m = n/(\Delta + 2)$.*

Proof. (a) Review of the above proof shows that Z is a 2-blocking set with $|Z| < (n - |Z|)\Delta$, so $|Z| < n\Delta/(\Delta + 1)$ unless each vertex in W has degree Δ and $|\Gamma_W(z)| = 1$ for each $z \in Z$. Since $|W| = \alpha(G)$, it follows that if $z_1, z_2 \in Z$ have a common neighbor in $w \in W$ then $z_1 z_2 \in E(G)$; otherwise, $W \setminus \{w\}) \cup \{z_1, z_2\}$ is an independent set with more than $|W|$ vertices. Hence each component of G is isomorphic to $K_{\Delta+1}$, and it follows that $G \cong mK_{\Delta+1}$ where $m = n/(\Delta + 1)$.

(b) Review of the above proof shows that Z is a 3-blocking set with $2|Z| < (n - |Z|)\Delta$, so $|Z| < n\Delta/(\Delta + 2)$ unless $W = V(G) \setminus Z$ is an independent set, each vertex $w \in W$ has degree Δ, and each vertex $z \in Z$ satisfies $|\Gamma_W(z)| = 2$. Let $w \in W$ be arbitrary, and consider $\langle \Gamma(w) \rangle$. We claim that Δ is even and $\langle \Gamma(w) \rangle \cong CP(\Delta/2)$. To prove this claim, we first note that if $z \in \Gamma(w)$ has degree $\Delta - 1$ in $\langle \Gamma(z) \rangle$ then it has degree $(\Delta - 1) + 2 > \Delta$ in G, a contradiction. Suppose $z \in \Gamma(w)$ has degree $\Delta - 3$ or less in $\langle \Gamma(w) \rangle$. Specifically, suppose that there are distinct vertices $z', z'' \in \Gamma(w)$ such that $zz' \notin E(G)$, and $zz'' \notin E(G)$. We may assume that $\Gamma_W(z) = \{w, w'\}$ where $w' \neq w$. If $\Gamma_W(z') = \{w, w''\}$ where $w'' \neq w'$, then $W' = (W \setminus \{w\}) \cup \{z, z'\}$ satisfies $|W'| > |W|$ and $P_3 \not\subset \langle W' \rangle$, a contradiction. Hence, we conclude that $\Gamma_W(z) = \Gamma_W(z') = \Gamma_W(z'') = \{w, w'\}$. But then $W'' = (W \setminus \{w, w'\}) \cup \{z, z', z''\}$ satisfies $|W''| > |W'|$ and $P_3 \not\subset \langle W'' \rangle$, a contradiction. It follows that $\langle \Gamma(w) \rangle \cong CP(\Delta/2)$ as claimed. Since each $z \in Z$ satisfies $|\Gamma_W(z)| = 2$ belongs to a subgraph of $\langle Z \rangle$ isomorphic to the cocktail-party graph $CP(\Delta/2)$, it follows that G is regular of degree Δ. Again suppose $zz' \notin E(G)$, so $\Gamma_W(z) = \Gamma_W(z') = \{w, w'\}$. Then $(W \setminus \{w, w'\}) \cup \{z, z'\}$ is an independent set. Thus z can play the role initially played by w. It follows that $\Gamma(w') = \Gamma(w)$ and the subgraph spanned by $\Gamma(w) \cup \{w, w'\}$ is isomorphic to $CP(\Delta/2 + 1)$. Clearly, such a subgraph is a component of G, for else some vertex has degree exceeding Δ, and repetition yields the fact that $G \cong mCP(\Delta/2 + 1)$ where $m = n/(\Delta + 2)$.

□

4 Grid Graphs

Let $GP(n) = P_n \times P_n$ and $GC(n) = C_n \times C_n$. We shall refer to $GP(n)$ as the *square grid graph*. Specifically, we shall take $GP(n)$ to be the graph with vertex set $V = \{(x, y) \mid 0 \leq x, y < n\}$ in which two vertices are adjacent if their indices agree in one coordinate and differ by exactly one in the other. The graph $GC(n)$ can be thought of as the $n \times n$ grid embedded on a torus.

Proposition 2. *If n is even, then $GP(n)$ has 2-blocking ratio $1/2$. The 3-blocking ratio is $1/2$ as well.*

Proof. The proof is practically the same as that for hypercubes. Let $Z = \{(x, y) \mid x+y \equiv 1 \pmod 2\}$. Then $|Z| = n^2/2$ and every edge of $GP(n)$ is incident with a vertex in Z, so Z is 2-blocking. In the other direction, note that $GP(n)$ has a 2-factor consisting of $n^2/4$ C_4's. Any 3-blocking set Z must contain at least two vertices from each of these C_4's, so the 3-blocking ratio is at least $1/2$. □

Proposition 3. *(a) The 4-blocking ratio of $G = GP(n)$ is at most $3/8$. (b) The 4-blocking ratio of $GC(n)$ is at least $3/8$. (c) The 4-blocking ratio of $GP(n)$ converges to $3/8$ as $n \to \infty$.*

Proof. (a) Note that for $G \cong GP(n)$ the set

$$Z = \{(x, y) \mid 0 \leq x, y < n, \quad x \pm y \equiv 0 \pmod 4\}$$

is 4-blocking, since each connected component of $G \setminus Z$ is isomorphic to $K_{1,4}$ or some subgraph thereof. Figure 1 shows Z (the darkened vertices) for the case of $n = 8$. In this case $|Z| = (3/8) \cdot 64 = 24$.

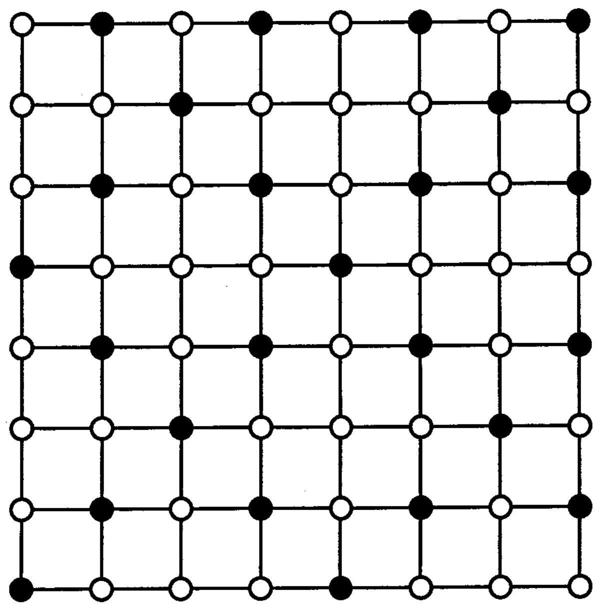

FIGURE 1. **4-Blocking Set for $GP(8)$**

(b) Note that $GC(n)$ has n^2 vertices and is regular of degree 4 so it has $2n^2$ edges. Suppose Z is a 4-blocking set. Then $G \setminus Z$ has $n^2 - |Z|$ vertices, at least $2n^2 - 4|Z|$ and contains no P_4. Since $G \setminus Z$ contains no P_4, each nontrivial component is isomorphic to $P_2, P_3, K_{1,3}$ or $K_{1,4}$.

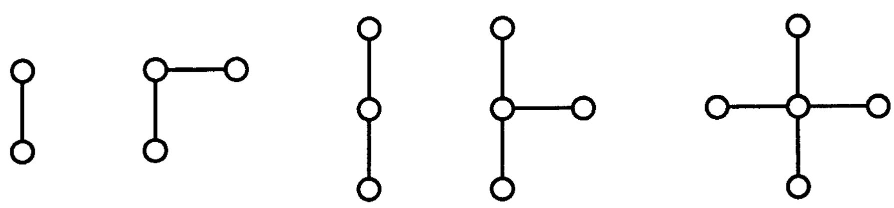

FIGURE 2. **Components of $G \setminus Z$**

These graphs have average degree 1, 4/3, 3/2, 8/5, respectively, and it follows that $G \setminus Z$ has average degree at most 8/5. Hence

$$\frac{2(2n^2 - 4|Z|)}{n^2 - |Z|} \leq \frac{8}{5},$$

which gives $|Z|/n^2 \geq 3/8$. (c) Clearly, as $n \to \infty$ the "edge effects" become negligible,

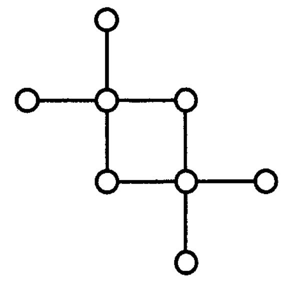

FIGURE 4.

To prove that the 6-blocking ratio of $G \cong GC(n)$ is at least $1/3$, it suffices to check that if Z is any 6-blocking set then each component of $G \setminus Z$ has average degree at most 2. Then $2(2n^2 - |Z|)/(n^2 - |Z|) \leq 2$, so $|Z|/n^2 \geq 1/3$.

Lemma 1. *If $p < r$, and p vertices are deleted from $GP(r)$, then the resulting graph contains a path of order $(r-p)^2 + p$.*

Proof. The vertices of $GP(r)$ fall into r rows and n columns. The deletion of any $p < r$ vertices leaves $r-p$ rows and $r-p$ columns intact. Then there is an obvious zig-zag path that uses the $r-p$ intact rows, and uses the leftmost and rightmost intact columns to go between these rows, as illustrated below. This gives a path with at least $(r-p)^2 + p$ vertices. □

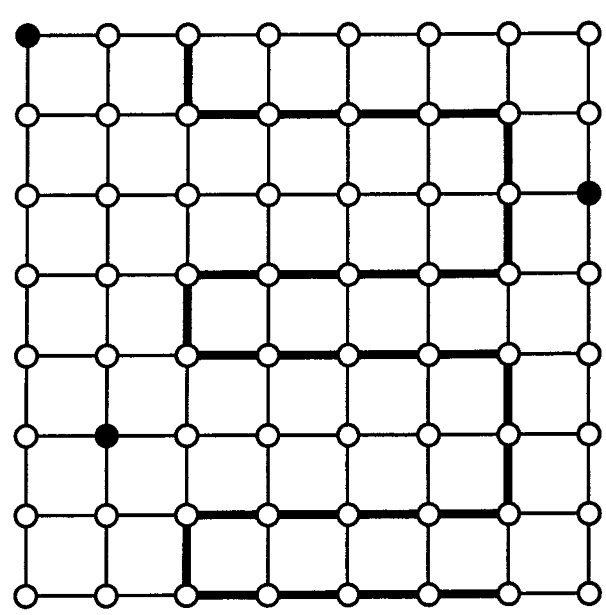

FIGURE 5. **Illustration of the Lemma**

Theorem 4. *For $k \geq 3$ and for all sufficiently large values of n, the k-blocking ratio of $GP(n)$ is between $1/(4\sqrt{k})$ and $\sqrt{2/k}$.*

Proof. First we prove that the k-blocking ratio is less than $\sqrt{2/k}$. For this purpose, we use the blocking set

$$Z = \{(x, y) \mid 0 \leq x, y < n, \;\; x \pm y \equiv 0 \pmod{m}\}, \qquad m = \lceil\sqrt{2k}\rceil.$$

For simplicity, assume first that m divides n. Then

$$\frac{|Z|}{n^2} = \begin{cases} \dfrac{2m-2}{m^2}, & m \text{ even}, \\ \dfrac{2m-1}{m^2}, & m \text{ odd}, \end{cases}$$

and the largest component of $GP(n) \setminus Z$ has $m^2/2 - m + 1$ vertices if m is even and $(m-1)^2/2$ vertices if m is odd. Examples of the largest component for two different values of m are shown in Fig. 6.

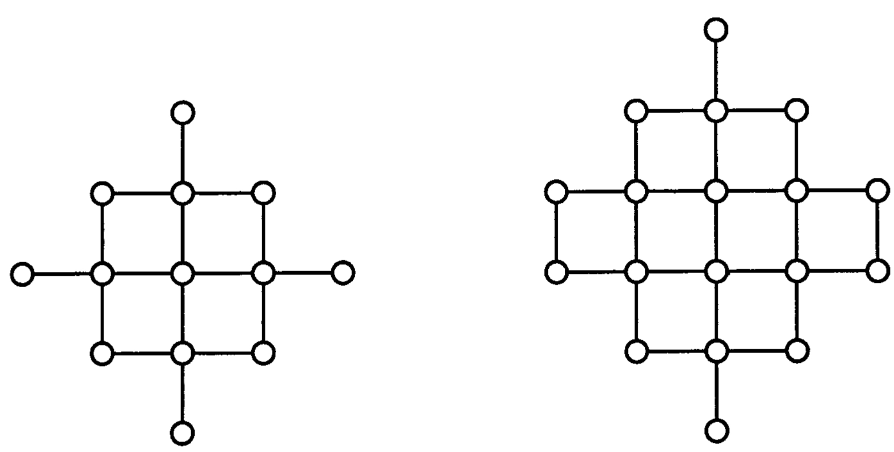

FIGURE 6. **Components of $GP(n) \setminus Z$**

With this choice, $|Z|/n^2 < 2/m < \sqrt{2/k}$ and $P_k \not\subset GP(n) \setminus Z$ since no component of $GP(n) \setminus Z$ has more than k vertices, and the largest component does not have a hamiltonian path. The same conclusion holds in case n is not divisible by m, since $|Z|/n^2 < 2/m$ still holds.

To prove the lower bound, we shall use Lemma 1. Set $d = \lceil\sqrt{k}\rceil = \sqrt{k} + \epsilon$ where $0 \leq \epsilon < 1$. For simplicity, first assume that $2d$ divides n. Then there are $(n/2d)^2$ copies of $GP(2d)$ in $G = GP(n)$. If $|Z| \leq n^2/(4\sqrt{k})$, then a simple averaging argument shows that in $GP(n)$ there is a copy of $GP(2d)$ having at most $(2d)^2/(4\sqrt{k}) = d^2/\sqrt{k}$ vertices in common with Z. By Lemma 1, in $GP(n)\setminus Z$ such a copy contains a path with at least

$$\left(2d - \frac{d^2}{\sqrt{k}}\right)^2 + \frac{d^2}{\sqrt{k}} = d^2\left(2 - \frac{d}{\sqrt{k}}\right)^2 + \frac{d^2}{\sqrt{k}}$$
$$= \frac{(k-\epsilon^2)^2}{k} + \sqrt{k} + 2\epsilon + \frac{\epsilon^2}{\sqrt{k}}$$
$$\geq k + \sqrt{k}$$

vertices. Now it is easy to see that the condition $(2d)|n$ can be removed provided n is sufficiently large. By continuity, we can choose $\delta > 0$ so that with $d^2/\sqrt{k}$ replaced by $d^2/\sqrt{k} + \delta$ in the above calculation, the final values is at least k. Then for all sufficiently large n, there exists a copy of $GP(2d)$ having at most $d^2/\sqrt{k} + \delta$ vertices in common with Z, and this gives the desired result. □

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