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On subgroups of amalgamated free products

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1. INTRODUCTION

In 1934 Kurosh (**9**) proved that 'a subgroup of a free product of groups is again a free product'. Many other proofs of this, and attempts to generalize it to amalgamated free products, have appeared (e.g. (**7**), (**1**), (**10**) and (**8**)). Recently the theory of groupoids has been applied to this area with increasing success. In 1966 Higgins (**6**) used groupoids to prove the generalization of Grushko's Theorem (**3**) due to Wagner (**14**).

We here apply groupoids to prove the following theorem:

THEOREM A. *Suppose*:

(1) $G = \Pi^*(G_\mu, \mu \in M; G_0)$ *is a free product of groups* G, μ *in an index set* M, *with* G_0 *amalgamated;*

(2) $K = \Pi^*(K_\mu, \mu \in M)$ *is a free product of groups over the same index set;*

(3) $f\colon G \to K$ *is a group homomorphism with* $f(G_\mu) \subset K_\mu$ *for each* μ; *and*

(4) H *is a subgroup of* G *such that* $f(H) = K$.

Then

(5) H *is expressible as* $\Pi^*(H_\mu, \mu \in M; H_0)$ *with* $f(H_\mu) \subset K_\mu$;

(6) H_0 *is generated as a subgroup of* G *by certain subgroups* $g_{0\nu} G_{0\nu} g_{0\nu}^{-1}$, *where* $G_{0\nu} \subset G_0$, $\{g_{0\nu}\} \subset \operatorname{Ker} f \subset G$;

(7) *Each* H_μ *is generated as a subgroup of* G *by certain subgroups* $g_{\mu\nu} G_{\mu\nu} g_{\mu\nu}^{-1}$, *where* $G_{\mu\nu} \subset G_\mu$, $\{g_{\mu\nu}\} \subset \{g_{0\nu}\} \subset \operatorname{Ker} f$, *together with certain elements* $g_1 g_2 g_3$, *where* $g_2 \in G$, g_1 *and* $g_3 \in \{g_{0\nu}\}$; *and*

(8) *Finally, if* $G_0 = \{1\}$ *then* $H_0 = \{1\}$ *and each* H *is the free product of the indicated subgroups together with the free group on the indicated elements.*

Notice that if $K = G_0 = \{1\}$, the Kurosh subgroup theorem follows. If $G_0 = \{1\}$, $K \neq \{1\}$, we have the following:

THEOREM OF HIGGINS (**6**). *Let* $G = \Pi^*(G_\mu, \mu \in M)$ *and* $K = \Pi^*(K_\mu, \mu \in M)$ *be free products. Let* $f\colon G \to K$ *satisfy* $f(G_\mu) \subset K_\mu$ *for all* μ, *and let* H *be a subgroup of* G *with* $f(H) = K$. *Then* $H = \Pi^*(H_\mu, \mu \in M)$ *with each* $f(H_\mu) \subset K_\mu$.

Higgins modifies an argument of Stallings (**13**) to deduce as a corollary of this

GRUSHKO'S THEOREM. *Let* $g\colon F \to \Pi^*(K_\mu, \mu \in M)$ *be a map of a free group onto a free product of groups. Then* F *is itself a free product* $F = \Pi^*(F_\mu, \mu \in M)$ *with* $g(F_\mu) \subset K_\mu$.

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2·3·2. As in the case of groups, we may find a canonical member of each equivalence class of words.

2·3·3. Let $a = [a_1 \ldots a_n]$, $n \geqslant 2$, be in reduced form. Then $a \neq [b]$ for any $b \in A_0$, and $a \neq [e]$ for any identity e.

It is critical to what follows that nowhere in section 2·3 is it required that A_0 or any A_μ be connected.

Higgins' free products of groupoids are now free products with various sets of identities amalgamated. The usual free product of groups is in our notation

$$\Pi^*(G_\mu, \mu \in M; \{1\})$$

but we shall omit the $\{1\}$. Note that $\Pi^*(G_\mu, \mu \in M; G_0)$ has only one identity 1, G_0 and each G_μ are connected, and G_0 is simply connected only if it is trivial.

We shall often omit the brackets in $[a_1 \ldots a_n]$. This will not cause confusion, for when we write $a_i = [a_i]$, $a_1 \ldots a_n$ is a product in $A = \Pi^*(A_\mu, \mu \in M; A_0)$ whose value is $[a_1 \ldots a_n]$. Note that now $A_0 \subset A_\mu \subset A$ for all $\mu \in M$, and the identities of A are just those of the various A_μ.

3. Some lemmas

3·1. *Simple constructions.*

LEMMA 3·1·1. *Let $B = \Pi^*(B_\mu, \mu \in M; B_0)$ and let $p: A \to B$ be a covering map. Denote $p^{-1}(B_\mu)$ by A_μ, $\mu \in M \cup \{0\}$. Then $A = \Pi^*(A_\mu, \mu \in M; A_0)$.*

Proof. If $\mu \neq \nu$, $A_\mu \cap A_\nu = p^{-1}(B_\mu) \cap p^{-1}(B_\nu) = p^{-1}(B_\mu \cap B_\nu) = p^{-1}(B_0) = A_0$. Thus we may construct $\Pi^*(A_\mu, \mu \in M; A_0)$. Since there are inclusions $A_\mu \subset A$, there is a map $f: \Pi^*(A_\mu, \mu \in M; A_0) \to A$. We show f is an isomorphism by producing an inverse. If $a \in A$, $p(a) \in B$ and may be written $p(a) = b_1 b_2 \ldots b_n$ with each b_i in some $B_{\mu(i)}$. $f(e(a)) = e(b_1)$, so there is a unique lifting of b_1 to $a_1 \in A_{\mu(1)}$ starting at $e(a)$. Since $f(e(a_1^{-1})) = e(b_1^{-1}) = e(b_2)$, b_2 lifts uniquely to $a_2 \in A_{\mu(2)}$, and so on. Then $a_1 \ldots a_n$ is the *unique* path starting at $e(a)$ and mapping to $f(a)$; that is, $a_1 \ldots a_n = a$. This gives a map $g: A \to \Pi^*(A_\mu, \mu \in M; A_0)$, namely $g(a) = [a_1 \ldots a_n]$, which is well-defined since an elementary equivalence on $b_1 \ldots b_n$ induces one on $a_1 \ldots a_n$. Since g is an inverse for f, 3·1·1 is proven.

Our next result is motivated by a simple case of Van Kampen's Theorem:

LEMMA 3·1·2. *Let $A = \Pi^*(A_\mu, \mu \in M; A_0)$ where A_0, each A_μ, and A are connected. Let e be an identity in A_0. Then $\pi(A, e) = \Pi^*(\pi(A_\mu, e), \mu \in M; \pi(A_0, e))$.*

Proof. Denote $\Pi^*(\pi(A_\mu, e), \mu \in M; \pi(A_0, e))$ by P. An element of P has a representative $a = a_1 \ldots a_n$ which is also a product in $\pi(A, e)$. Since an elementary equivalence in P induces one in A, the resulting map $p: P \to \pi(A, e)$ is well defined. We show that this map is one-to-one and onto.

Onto. Given $a = a_1 \ldots a_n \in \pi(A, e)$, we shall find a pre-image in P. Here a_i is in some $A_{\mu(i)}$ and $e(a_1) = e(a_n^{-1}) = e$. For each a_i, let r_i be a path in A_0 from e to $e(a_i)$; this exists since A_0 is connected. Now $a = (a_1 r_2^{-1})(r_2 a_2 r_3^{-1}) \ldots (r_{n-1} a_{n-1} r_n^{-1})(r_n a_n)$ is an element of P which maps to a under $p: P \to \pi(A, e)$.

One-to-one. Suppose $a = a_1 \dots a_n$ and $c = c_1 \dots c_m$ are in P, so that each a_i and c_j lie in some $\pi(A_\mu, e)$. Suppose $p(a) = p(c)$ in A; that is, $a_1 \dots a_n c_m^{-1} \dots c_1^{-1}$ may be reduced to an identity by elementary equivalences in A. Now 2·3·3 tells us that at no step in this process do we have a nontrivial reduced word. Hence at each step we may merely eliminate an identity, or multiply two terms together. But these steps can be carried out as easily in P as in A, so $a = c$ in P, concluding the proof.

3·2. *A method of restructuring products.* When a product structure is lifted to a covering groupoid, as in 3·1·1, the amalgamated subgroupoid of the cover is rarely connected, hindering application of 3·1·2. It is thus useful to be able to enlarge the amalgamated subgroupoid until it is connected. The results here are closely related to Stallings' (13) 'Construction in the case of binding ties'.

LEMMA 3·2·1. *Suppose* $A = \Pi^*(A_\mu, \mu \in M; A_0)$, $s \in \bigcup A_\mu \setminus A_0$; *let* A'_μ *be the subgroupoid of* A *generated by* $A_\mu \cup \{s\}$, *for* $\mu \in M \cup \{0\}$. *Then* $A = \Pi^*(A'_\mu, \mu \in M; A'_0)$.

Proof. We first show that if $\mu \neq \nu$, $A'_\mu \cap A'_\nu = A'_0$. Clearly $A_0 \subset A_\mu \subset A'_\mu$ and $s \in A'_\mu$, so $A'_0 \subset A'_\mu$. Similarly $A'_0 \subset A'_\nu$, so $A'_0 \subset A'_\mu \cap A'_\nu$. Suppose that this fails to be an equality: suppose $q_0 \in A'_\mu \cap A'_\nu \setminus A'_0$. Write

$$q_0 = c'_1 \dots c'_j, \quad c'_i \in A_\mu \cup \{s, s^{-1}, e(s), e(s^{-1})\} \subset A_\mu \cup A_{\mu'}, \quad s \in A_{\mu'}.$$

Multiply terms together to write $q_0 = [c_1 \dots c_n]$ in reduced form in A. Now no c_i is an identity (unless $n = 1$, an easy special case), and the c_i lie alternately in $A_\mu \setminus A_0$ and in $(A_{\mu'} \cap A'_0) \setminus A_0$. Surely some $c_i \notin A'_0$ (otherwise $q_0 \in A'_0$ contrary to hypothesis). Suppose c_{r+1} through c_n lie in A'_0, $c_r \notin A'_0$, so $c_r \in A_\mu \setminus A_0$. Now let $q = [c_1 \dots c_r] \in A'_\mu \cap A'_\nu \setminus A'_0$. Again write q in reduced form in A as $q = [d_1 \dots d_m]$ so that the d_i lie alternately in $A_\nu \setminus A_0$ and in $(A_{\mu'} \cap A'_0) \setminus A_0$.

Now $e(q) = qq^{-1} = [c_1 \dots c_r d_m^{-1} \dots d_1^{-1}]$, and by 2·3·3, d_m must lie in A_μ along with c_r. Also $c_r d_m^{-1}$ must lie in A_0 (for we can reduce no further unless this term lies in the same factor as the terms around it, $A_{\mu'}$; and $A_{\mu'} \cap A_\mu = A_0$). (If $\mu = \mu'$, interchange ν and μ; at least one of them is not μ'.) (If $r = 1$ or $m = 1$ $c_r d_m^{-1}$ may be an identity. Then $c_r = d_m \in (A_\mu \setminus A_0) \cap (A_\nu \cup A_{\mu'} \cap A'_0 A_0) \subset A'_0$, contradicting $c_r \notin A'_0$.) But since

$$c_r d_m^{-1} \in A_0 \quad \text{and} \quad d_m \in A_{\mu'}, \quad d_m \in A_\mu \cap [(A_\nu \setminus A_0) \cup (A_{\mu'} \cap A'_0)] \subset A'_0.$$

Now $d_m \in A'_0$ and $c_r d_m^{-1} \in A'_0$, so $c_r \in A'_0$, a contradiction, and we have shown that $A'_\mu \cap A'_\nu = A'_0$.

We now denote $\Pi^*(A'_\mu, \mu \in M; A'_0)$ by A', and proceed to show $A' = A$. An element of A may be written in the form $h = [h_1 \dots h_n]$, $h_i \in A_{\mu(i)}$. An element of A' may be written as an equivalence class $h' = \langle h'_1 \dots h'_m \rangle$, $h'_i \in A'_{\mu'(i)}$. Finally, each of the h'_i is a member of A and thus may be written as $h'_i = [h_{i,1} \dots h_{i,r(i)}]$, $h_{i,j} \in A_{\mu''(i,j)}$. Now define a map $f: A \to A'$ by $f([h_1 \dots h_n]) = \langle h_1 \dots h_n \rangle$, $h_i \in A_{\mu(i)} \subset A'_{\mu(i)}$. f is a well-defined groupoid map. Define $f': A' \to A$ by $f(\langle h'_1 \dots h'_m \rangle) = [h_{1,1} \dots h_{1,r(1)} h_{2,1} \dots h_{m,r(m)}]$ where $[h_{i,1} \dots h_{i,r(i)}]$ is any word representing $h'_i \in A'_{\mu'(i)}$ as a member of $A = \Pi^*(A_\mu, \mu \in M; A_0)$.

Now f' is well defined since

(1) Any two representatives of h'_i are equivalent in A.

(2) An identity in A' is one in A.

(3) If $h_i' h_{i+1}' = h_i^*$ is an elementary equivalence in A' (so that all three lie in some A_μ'), then $[h_{i,1} \dots h_{i,r(i)} h_{i+1,1} \dots h_{i+1,r(i+1)}]$ is a representation of h_i^* in A, and any other representation of h_i^* is equivalent to it.

Thus any elementary equivalence in A' induces an equivalence in A; so f' is well defined. Since f and f' are inverses, A is isomorphic to A', and 3·2·1 is proven.

Having enlarged A_0 by 3·2·1, we have a new representation for A and can again apply 3·2·1 to it. Proceed by transfinite induction: the limit steps present no real difficulty, since if an element of A is generated by A_μ and an infinite set S it is in fact generated by A_μ and finitely many elements of S. The argument appears in detail in (11). We reach eventually

THEOREM 3·2·2. *Suppose* $A = \Pi^*(A_\mu, \mu \in M; A_0)$. *Suppose S is a subset of A well ordered by* $<$, *and that if* $s \in S$, *then s is generated by* $A_\mu \cup \{s' \in S | s' < s\}$ *for some* μ. *Let A_μ' denote the subgroupoid of A generated by* $A_\mu \cup S$, *for* $\mu \in M \cup \{0\}$. *Then* $A = \Pi^*(A_\mu', \mu \in M; A_0')$.

4. PROOF OF THEOREM A

4·1. A (5). While certain of the following constructions work in more general cases, we now mainly restrict attention to the situation of Theorem A, and apply the results of section 3 to express H as an amalgamated free product.

Let $G = \Pi^*(G_\mu, \mu \in M; G_0)$ and H be a subgroup of G. By 2·2·3, since G is a groupoid with $\pi(G, 1) = G$, there are a groupoid C, an identity $e_0 \in C$, and a covering map p: $C \to G$ with $p(e_0) = 1$, such that p is an isomorphism from $\pi(C, e_0)$ onto H. By 3·1·1, $C = \Pi^*(C_\mu, \mu \in M; C_0)$ where $C_\mu = p^{-1}(G_\mu)$, $\mu \in M \cup \{0\}$.

We make conventions regarding base-points in C. The e_0 referred to above is the base-point of C and C_0. Each component $C_{0\nu}$ of C_0 will have a base-point (denoted $e_{0\nu}$) with $e_0 = e_{00} \in C_{00}$ the base-point of its component, and $e_{0\nu} \in p^{-1}(1)$ otherwise (each $C_{0\nu}$ intersects $p^{-1}(1)$, since it must contain some identity). Each component $C_{\mu\nu}$ of C_μ contains at least one component of C_0 (since C is connected); choose an $e_{\mu\nu}$ from the possible $e_{0\nu}$'s. For any subset X of C, denote the subgroupoid generated by $C_\mu \cup X$ by $C_\mu(X)$, and take as its base-point some $e_{\mu\nu}$ contained in it. We will often denote the base-point of a groupoid by *, when the groupoid involved is clear.

The following theorem combines two theorems of Stallings (13).

THEOREM 4·1·1. *Suppose* $C = \Pi^*(C_\mu, \mu \in M; C_0)$ *and* $B = \Pi^*(B_\mu, \mu \in M; B_0)$ *are amalgamated free products of groupoids,* $f: C \to B$ *such that* $f(C_\mu) \subset B_\mu$. *Suppose:*

(1) *C, B, and B_0 are connected, C_0 is disconnected.*

(2) $f: \pi(C, *) \to \pi(B, *)$ *is onto.*

(3) *If a is a loop in C_μ,* $f(a) \in B_0$, *and $f(a)$ is not an identity, then* $a \in C_0$.

Then there exists a $\mu' \in M$ *and a path* $s \in C_{\mu'}$, *such that $e(s)$ and $e(s^{-1})$ are base-points of distinct components of C_0, and* $f(s) \in B_0$.

Proof. Let $p \in C$ be a path connecting base-points of two components of C_0; then the product $e_{01} p e_{02}$ is defined in C. $f(p)$ is a path in B with end-points in B_0. There is a path p' in B_0 from $f(e_{02})$ to $f(e_{01})$. $f(p)p'$ is a loop in B at $f(e_{01})$. Since the map restricted to fundamental groups is onto, there is a loop $r \in \pi(C, e_{01})$ with $f(r) = f(p)p'$. Let $q = p^{-1}r$; then q is a path from e_{02} to e_{01} and $f(q) = f(p^{-1})f(p)p' = p' \in B_0$.

Next write $q = q_1 \dots q_n$ in reduced form: no q_i an identity, each q_i in a $C_{\mu(i)}$, $\mu(i) \neq \mu(i+1)$. Write at each juncture $q_i(r_i r_i^{-1}) q_{i+1} = (q_i r_i)(r_i^{-1} q_{i+1})$ for $q_i q_{i+1}$, where $r_i \in C_0$ runs from $e(q_{i+1})$ to the base-point of that component of C_0, so that each q_i has as end-points base-points $e_{0\nu}$.

If any of the q_i is a loop with $f(q_i)$ an identity in B, write q without that q_i (preserving $f(q) \in B_0$) and again put it in reduced form. Note that if ever $q = q_1$, the theorem is proven.

Now if $q = q_1 \dots q_n$ denotes the finally reduced expression, and if q_i is a loop, then $f(q_i)$ is a non-trivial loop and by (3) cannot lie in B_0. Now write $f(q) = f(q_1) \dots f(q_n) = p' \in B_0$. By 2·3·3, some $f(q_i)$ lies in B_0. That q_i cannot be a loop; so it is the path s demanded by the theorem.

Now consider the case of Theorem A. Here $f: \Pi^*(G_\mu, \mu \in M; G_0) \to \Pi^*(K_\mu, \mu \in M)$. $p: C \to G$ is a covering map, and $fp: C \to K$ satisfies the hypotheses of 4·1·1 if C_0 is disconnected. Let s denote the path found by 4·1·1. Now by 3·2·1,

$$C = \Pi^*(C_\mu(\{s\}), \mu \in M; C_0(\{s\})).$$

Further, since $fp(s) \in K_0 = \{1\}$, $s \in \ker fp$ and also $fp(C_\mu(\{s\})) \subset K_\mu$. Hence, if $C_0(\{s\})$ is disconnected, we may apply 4·1·1 and 3·2·1 again. We may continue by transfinite induction, applying at each stage 4·1·1 and 3·2·2, finally obtaining

PROPOSITION 4·1·2. *Given a groupoid* $C = \Pi^*(C_\mu, \mu \in M; C_0)$ *and a group*

$$K = \Pi^*(K_\mu, \mu \in M),$$

$fp: C \to K$ *with* $fp(C_\mu) \subset K_\mu$, C *connected*, $fp: \pi(C, *) \to K$ *onto, and* C_0 *disconnected, there exists an* $S \subset C$ *with the following properties:*

(1) S *is well ordered, with order denoted* $<$. *If* $s \in S$, *and* $C_\mu(S_s)$ *is generated by* $C_\mu \cup \{s' \in S | s' < s\}$, *then* $s \in C_\mu(S_s)$ *for some* μ.

(2) *The end-points of* $s \in S$ *lie in* C_0 *and are base-points of distinct components of* $C_0(S_s)$.

(3) $fp(S) = \{1\}$.

(4) $C_0(S)$ *is connected.*

COROLLARY 4·1·3. *Under the hypotheses of* 4·1·2,

$$\pi(C, *) = \Pi^*(\pi(C_\mu(S), *), \mu \in M; \pi(C_0(S), *))$$

by 3·1·2. *Under the hypotheses of Theorem A, letting* $H_\mu = p\pi(C_\mu(S), *)$,

$$H = p\pi(C, *) = \Pi^*(H_\mu, \mu \in M; H_0),$$

since p *is an isomorphism of* $\pi(C, *)$. *Also,* $f(H_\mu) = fp\pi(C_\mu(S), *) \subset fp(C_\mu(S)) \subset K_\mu$.

We have thus established conclusion (5) of Theorem A.

4·2. A(6). A clear intuitive implication of (1) and (2) of 4·1·2 is that S is 'treelike'. Let Q be the subgroupoid of C generated by S.

LEMMA 4·2·1. *Q is connected and simply connected.*

Proof. The identities of Q are precisely the base-points $e_{0\nu}$. $C_0(S)$ is connected, so it contains a path between any two base-points e_{01} and e_{02}. But then such a path can be built using only factors from S, as the factors in C_0 never run from one component of C_0 to another. Hence Q is connected.

To show Q is simply connected, suppose $s_1 \dots s_n$ to be a non-trivial loop of minimal length, $n > 1$, since an s_i can be a loop only if it is an identity. No s_i, s_j can be equal for $i < j$, for then $s_i \dots s_{j-1}$ would be a shorter loop. Similarly no $s_i = s_j^{-1}$. Thus some $s_i^{\pm 1}$ is the greatest (for the order $<$ of S) of $s_1^{\pm 1}, s_2^{\pm}, \dots s_n^{\pm 1}$. Hence $s_{i+1} \dots s_n s_1 \dots s_{i-1} \in C_0(S_{s_i})$, and thus the two end-points of s_i lie in the same component of $C_0(S_{s_i})$, contradicting (2) of 4·1·2. Thus Q contains no non-trivial loop.

It follows that if e_{01} and e_{02} are base-points of two components, there is a unique path in Q connecting them.

LEMMA 4·2·2. *Under the hypotheses of* 4·1·2, *if $c \in \pi(C_0(S), *)$, then c may be written in the form $c = (q_1 c_1 q_1^{-1})(q_2 c_2 q_2^{-1}) \dots (q_n c_n q_n^{-1})$ where each $c_i \in \pi(C_{0\nu}, e_{0\nu})$, $\nu = \nu(i)$, and each q_i is the unique path in q from $e_{00} = e_0$ to $e_{0\nu}$.*

Proof. Since $c \in \pi(C_0(S), *)$, write c as $d_1 \dots d_m$, $d_i \in C_0$ or $d_i \in S$ or $d_i^{-1} \in S$. Group terms so $c = q_1' c_1 q_2' c_2 \dots c_n q_{n+1}'$, $c_i \in C_0$, $q_i' \in Q$. Now each c_i starts and ends at a base-point $e_{0\nu}$, so $c_i \in \pi(C_{0\nu}, e_{0\nu})$. Let q_i be the unique path in Q from e_0 to $e_{0\nu}$. Now

$$\begin{aligned} c &= q_1' c_1 (q_1^{-1} q_1) q_2' (q_2^{-1} q_2) c_2 \dots (q_n^{-1} q_n) c_n q_{n+1}' \\ &= (q_1' c_1 q_1^{-1})(q_1 q_2' q_2^{-1})(q_2 c_2 q_2^{-1}) \dots (q_n c_n q_{n+1}') \\ &= (q_1 c_1 q_1^{-1})(q_2 c_2 q_2^{-1}) \dots (q_n c_n q_n^{-1}) \end{aligned}$$

by the simple-connectedness of Q, completing the proof.

PROPOSITION 4·2·3. *Under the hypotheses of Theorem A, H_0 is generated by certain subgroups $g_{0\nu} G_{0\nu} g_{0\nu}^{-1}$, with each $G_{0\nu} \subset G_0$ and each $g_{0\nu} \in \operatorname{Ker} f$.*

Proof. $H_0 = p\pi(C_0(S), *)$. But $\pi(C_0(S), *)$ is generated by the various $q_\nu \pi(C_{0\nu}, e_{0\nu}) q_\nu^{-1}$, so H_0 is generated by the $p(q_\nu) p\pi(C_{0\nu}, e_{0\nu}) p(q_\nu^{-1})$. Further, p carries $\pi(C_{0\nu}, e_{0\nu})$ isomorphically onto a subgroup of G_0, since $p|\pi(C_0(S), e_{0\nu})$ is one-to-one and

$$\pi(C_{0\nu}, e_{0\nu}) \subset \pi(C_0(S), e_{0\nu}).$$

Finally, $g_{0\nu} \in \operatorname{Ker} f$ since $g_{0\nu}$ lies in $p(Q)$ and $fp(Q) = 1$ by Theorem 4·1·2.

This establishes (6) of Theorem A. The machinery may in fact lend itself to a stronger result. For instance, Karrass and Solitar (8) express a subgroup of $(A*B; U)$ as a free product with amalgamated subgroup and show that the amalgamated subgroup may be expressed as a 'tree product' of certain subgroups. It would seem plausible to hope that $\pi(C_0(S), *)$, generated by the groups $\pi(C_{0\nu}, *)$ conjugated by elements of the tree Q, is in fact such a 'tree product'.

4·3. A (7). Establishing that H_μ has the desired structure is only slightly more complex. Since H_μ is isomorphic to $\pi(C_\mu(S), *)$, we study that. For each component $C_{\mu\nu}$ of C_μ, choose a path $q_{\mu\nu} \in Q$ from e_0 to $e_{\mu\nu}$. Then $q_{\mu\nu} \pi(C_{\mu\nu}, e_{\mu\nu}) q_{\mu\nu}^{-1} \subset \pi(C_\mu(S), *)$.

Note that there may be several paths running from e_0 to $C_{\mu\nu}$, since $C_{\mu\nu}$ may contain more than one component of C_0. Let us denote such a path in Q by $q_{\mu\delta}$. For each $q_{\mu\delta}$ such that $q_{\mu\nu}^{-1}q_{\mu\delta}\notin C_\mu$, choose a path $c_{\mu\delta}$ in $C_{\mu\nu}$ running from $e(q_{\mu\nu}^{-1})$ to $e(q_{\mu\delta}^{-1})$, so that $q_{\mu\nu}c_{\mu\delta}q_{\mu\delta}^{-1}$ is a loop in $C_\mu(S)$.

LEMMA 4·3·1. *$\pi(C_\mu(S),*)$ is generated by the groups $q_{\mu\nu}\pi(C_{\mu\nu},e_{\mu\nu})q_{\mu\nu}^{-1}$ together with the elements $q_{\mu\nu}c_{\mu\delta}q_{\mu\delta}^{-1}$.*

Proof. $c\in\pi(C_\mu(S),*)$ may be written $c = q_0'c_1q_1'c_2\ldots c_nq_n'$, with $c_i\in C_\mu$, $q_1'\in Q$. Insert terms $q_i^{-1}q_i$, where q_i starts at e_0, so that

$$\begin{aligned} c &= q_0'c_1(q_1^{-1}q_1)q_1'c_2(q_2^{-1}q_2)\ldots(q_{n-1}^{-1}q_{n-1})(q_{n-1}')c_nq_n' \\ &= (q_0'c_1q_1^{-1})(q_1q_1'c_2q_2^{-1})\ldots(q_{n-1}q_{n-1}'c_nq_n'). \end{aligned}$$

That is, c is generated by terms of the form $q_jc_iq_k^{-1}$ where $c_i\in C_\mu$ and q_j and q_k start at e_0. But all such terms are generated in the desired way: for if $q_j^{-1}q_{\mu\nu} = c_0\in C_{\mu\nu}$, then $q_jcq_k^{-1} = q_{\mu\nu}(c_0^{-1}c)q_k^{-1}$; while if $q_j^{-1}q_{\mu\nu}\notin C_{\mu\nu}$, one of the generators is an element $q_{\mu\nu}c_jq_j^{-1}$, $c_j\in C_{\mu\nu}$, and $q_jcq_k^{-1} = (q_{\mu\nu}c_jq_j^{-1})^{-1}(q_{\mu\nu}c_jcq_k)$. The q_k may be accounted for similarly.

PROPOSITION 4·3·2. *Under the hypotheses of Theorem A, H_μ is generated by certain subgroups $g_{\mu\nu}G_{\mu\nu}g_{\mu\nu}^{-1}$ together with certain elements $g_1g_2g_3$, where $G_{\mu\nu}\subset G_\mu$, $g_2\in G_\mu$, and all $g_{\mu\nu}, g_1, g_3\in\ker f$.*

Proof. Here $G_{\mu\nu} = p\pi(C_{\mu\nu},e_{\mu\nu})$, $g_{\mu\nu} = p(q_{\mu\nu})$, and $g_1g_2g_3 = p(q_{\mu\nu}c_{\mu\delta}q_{\mu\delta}^{-1})$. In particular, $g_2 = p(c_{\mu\delta})\in p(C_{\mu\nu}) = G_{\mu\nu}\subset G_\mu$.

This completes (7) of Theorem A.

4·4. *A* (8). It remains to show that if $G_0 = \{1\}$, the situation reduces to the Kurosh theorem. Since then H_0 is generated by groups $g_{0\nu}\{1\}g_{0\nu}^{-1} = \{1\}$, $H_0 = \{1\}$. Throughout 4·4, we suppose $G_0 = \{1\}$. Recall that S is well-ordered and that for any $s\in S$, S_s denotes $\{s'\in S | s' < s\}$.

LEMMA 4·4·1. *C_0 is simply connected, and for any s, $C_0(S_s)$ is simply connected.*

Proof. C_0 can contain no non-trivial loop since $p(C_0)\subset G_0 = \{1\}$ and thus $p(l) = p(e(l))$ for any loop l, so $l = e(l)$ is an identity. 4·2·2 implies that $C_0(S)$ is simply connected when C_0 is, and since $C_0(S_s)\subset C_0(S)$, $C_0(S_s)$ is simply connected.

LEMMA 4·4·2. *Let $c = q_0c_1q_1\ldots c_nq_n$ be a loop in $C_\nu(S)$ for some ν, where each $c_i\in C_\nu$, and each q_i is generated by elements in $S\setminus C_\nu$, and no c_i or q_i is an identity (except perhaps q_0, q_n). Then c is not an identity in C.*

Proof. Write out c as a product of elements of C_ν and S, where $s\in S$ appears as s or as s^{-1}. Define the *rank* of c to be the greatest s (in the order on S) which appears (where the expression for c is chosen to minimize this) and 0 if no s appears. Now every element $c\in C_\nu(S)$ has a rank in $S\cup\{0\}$; we prove the lemma by induction.

If rank $(c) = 0$, $c = c_1$ and we are done, since c_1 is not an identity. Suppose now that rank $(c) = s$, and that the lemma holds for all c' of lesser rank. Surely $s\notin C_\nu(S_s)$, for if not, c can be written without using s, and rank $(c) < s$. So, by 4·1·2, $s\in C_{\nu'}(S_s)$ for some $\nu'\neq\nu$. Write c in reduced form relative to $C = \Pi^*(C_\mu(S_s),\mu\in M; C_0(S_s))$:

$$c = d_0s^{\pm1}d_1s^{\pm1}\ldots s^{\pm1}d_m,\quad d_i\in C_\nu(S_s),\quad s^{\pm1}\in C_{\nu'}(S_s).$$

Now if any d_i fails to be a loop, that d_i (or perhaps $d_1 d_0$, if $m = 1$) connects $e(s)$ and $e(s^{-1})$, and thus cannot lie in $C_0(S_s)$. If every d_i is a loop, some d_i is not an identity (since some c_i is not an identity) and thus cannot lie in $C_0(S_s)$ (which contains no non-trivial loops). Hence some $d_i \in C_\nu(S) \setminus C_0(S)$, and by 2·3·3, c is not an identity.

PROPOSITION 4·4·3. *Under the conditions of Theorem A, with $G_0 = \{1\}$, H_μ is freely generated by the groups $p(q_{\mu\nu}) p(\pi(C_{\mu\nu}, e_{\mu\nu})) p(q_{\mu\nu}^{-1})$ and the elements $p(q_{\mu\nu} c_{\mu\delta} q_{\mu\delta}^{-1})$.*

Proof. We show that $\pi(C_\mu(S), *)$ is a suitable free product in C. It is generated by the proper things, by 4·3·2. If it is not freely generated, then there is some expression $r = (q_1 c_1 q_1')\ldots(q_n c_n q_n')$ which is reduced and non-trivial in the free product, but an identity in $C_\mu(S)$.

However, r can collapse to an identity when reduced to the form of 4·4·2 only if all of the $q_i' q_{i+1}$ collapse to identities. The only way elements of the big free product can meet this condition is if $q_{i+1} c_{i+1} q_{i+1}'$ is one of the elements $q_{\mu\nu} c_{\mu\delta} q_{\mu\delta}^{-1}$ and $q_i c_i q_i'$ comes from the group $q_{\mu\nu} \pi(C_{\mu\nu}, e_{\mu\nu}) q_{\mu\nu}^{-1}$ (otherwise, r would not be reduced in the free product generated by the groups and elements). But then

$$(q_i c_i q_i')(q_{i+1} c_{i+1} q_{i+1}') = q_{\mu\nu} c_i c_{\mu\delta} q_{\mu\delta}^{-1}$$

which cannot vanish since c_i is a loop and $c_{\mu\delta}$ is a path. Further, $q_{\mu\nu} \neq (q_{i-1}')^{-1}$ and $q_{\mu\delta}^{-1} \neq q_{i+2}$; so no additional reductions interact with this one. Hence, r does not vanish when written as in 4·4·2, so r cannot be an identity in $C_0(S)$.

This completes the proof of Theorem A.

5. REMARKS

5·1. Theorem 3·2·2 has a simple corollary obtained by replacing groupoids with groups. One implication is that if $G = \Pi^*(G_\mu, \mu \in M; G_0)$, it is easy to find an expression $G = \Pi^*(G_\mu', \mu \in M; G_0')$ with $G_\mu' \supset G_\mu$, $G_0' \supset G_0$. It might be of some interest to find 'finer' factorizations, i.e. $G = \Pi^*(G_\mu'', \mu \in M; G_0'')$ with $G_\mu'' \subset G_\mu$. Do there in general exist minimal factorizations? If $H \subset G$ and G has a minimal factorization, can one be found for H? In a later paper, the author hopes to explore implications of a converse of 3·2·2: if G_μ and G_μ' are related as above, G has a subset S such that G_μ, S, and G_μ' satisfy 3·2·2.

5·2. Theorem 4·1·1 has a limited amount of extra generality. The following is an easy corollary.

PROPOSITION 5·2·1. Let *G and K be amalgamated free products of groups (K is not required to be a free product), $f: G \to K$ a factor-preserving map, and H a subgroup of G with $f(H) = K$. Then either $H = \Pi^*(H_\mu, \mu \in M; H_0)$ with each $H_\mu \subset G_\mu$, or there is a g in some $G_\mu \setminus G_0$ with $f(g) \in K_0$.*

Proof. Let C be a connected covering groupoid of G with $\pi(C, *) \cong H$. If C_0 is connected, H may be factored in the desired way by 3·1·2. If C_0 is disconnected, apply 4·1·1. If hypothesis (3) fails, there is a loop a such that $p(a)$ is the desired g. If (3) holds, the theorem produces an s such that $p(s)$ is the desired g.

5·3. An alternative way to obtain $H_\mu \subset G_\mu$ is to enlarge G_μ, by applying 3·2·2.

PROPOSITION 5·3·1. *Let* $G = \Pi^*(G_\mu, \mu \in M; G_0)$ *and* $K = \Pi^*(K_\mu, \mu \in M)$ *be an amalgamated free product and a free product of groups. Let* $f: G \to K$ *satisfy* $f(G_\mu) \subset K_\mu$ *for all* μ. *Suppose* H *is a subgroup of* G *such that* $f(H) = K$. *Then there are factorizations* $G = \Pi^*(G_\mu(R), \mu \in M; G_0(R))$ and $H = \Pi^*(H_\mu, \mu \in M; H_0)$ *with* $H_\mu \subset G_\mu(R)$ *and*

$$f(G_\mu(R)) \subset K_\mu.$$

$G_\mu(R)$ *denotes the subgroup of* G *generated by* $G_\mu \cup R$, *where the set* R *may be chosen to be well ordered in such a way that if* $r \in R$, r *is generated by the elements preceeding* r *in* R, *together with elements of some one* G_μ.

Proof. Construct the covering $p: C \to G$ with $p\pi(C, *) = H$. If C_0 is connected, we are done, with R empty. If not, find the subset S given by 4·1·2. Now $p(S) = R \subset G$ satisfies the conditions of 3·2·2, yielding the desired expression for G. Since $f(R) = \{1\}$, and $H_\mu = p\pi(C_\mu(S), *)$, the result follows easily.

This proposition yields helpful results only if $G \neq G_0(R)$. One way to guarantee this is to have $K \neq \{1\}$, since then $f(G_0(R)) = \{1\} \neq K = f(G)$. An alternative way presents itself, for instance, if H is of finite index n in G; C_0 then has at most n components, so S and R are generated by at most $n-1$ elements.

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