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FREE PRODUCTS OF TOPOLOGICAL GROUPS WITH EQUAL UNIFORMITIES, II

BY

EDWARD T. ORDMAN (LEXINGTON, KENTUCKY)

1. Introduction. The free product of a collection of topological groups is known to exist, and to have, as its underlying abstract group, the algebraic free product of the underlying abstract groups of the factors [11]. Morris [8] observed, using results of Hulanicki [4], that a free product of maximally almost periodic (M.A.P.) groups is again M.A.P. (a topological group is called *M.A.P.* if there is a continuous one-to-one homomorphism of it into a compact Hausdorff group). In [11] we found it convenient to work with free products of locally invariant Hausdorff groups, where we call a group *locally invariant* if it meets (one of) these three equivalent conditions (see [1], [3], [5] and [12]):

(a) Every neighborhood N of the identity $e \in G$ contains a neighborhood U of e such that $gUg^{-1} \subset U$ for all $g \in G$.

(b) The right and left uniformities of G are equal.

(c) The topology of G can be constructed using the set of continuous 2-sided invariant pseudometrics on G.

This situation raises the question whether a free product of locally invariant groups is locally invariant, and we are able to see in Section 3 that this is false in a large class of cases. This leads to the construction of a large class of M.A.P. groups which are connected but not locally invariant. These groups are not locally compact, and thus motivate the proof, in Section 4, that free products of sufficiently nice factors are not locally compact, and that no free products of non-trivial groups are compact. In particular, free products of connected M.A.P. groups are not locally compact; this is of interest since such a product maps monomorphically into a compact group (see [4] and [8]). It can be of interest to contrast these pathologies with the comparatively well-behaved situation of [9].

2. Preliminaries. Any work in this area relates to the following well-known theorem (see [2], [3], [6] and [10]):

THEOREM A. *If G is a connected, locally compact, Hausdorff group, the following are equivalent:*

(a) *G is M.A.P.*

(b) *G is locally invariant.*

(c) *G is a direct product $K\times V$ of a compact group K and a vector group V.*

We also need the following results not already quoted:

PROPOSITION 1. *A free product of connected groups is connected.*

Proof. The component of the product containing the identity contains each factor, and thus contains the subgroup generated by their union, which is the entire group.

PROPOSITION 2. *If F is a free product of topological groups G_a, $a\in A$, and F is Hausdorff, then each G_a is closed in F.*

Proof. Define $k_\beta\colon G_\beta\to G_a$ to be the identity map if $\beta=a$, and the map $G_\beta\to\{e\}$ if $\beta\neq a$. These maps extend to a continuous homomorphism $k\colon F\to G_a$. Now, if $\{g_\delta\}_{\delta\in\Delta}$ is a net in $G_a\subset F$ converging to $g\in F$, then $g=\operatorname{Lim} g_\delta=\operatorname{Lim} k(g_\delta)=k(g)\in G_a$ and G_a is closed.

The topology τ. The topology of the free product F of topological groups G_a is rather intractable. We recall, briefly, a topology τ imposed on the abstract group F in [11], defined whenever each G_a is locally invariant. For each collection $\{\varrho_a\}$ of invariant pseudometrics on the G_a and for $\varepsilon>0$, define $N_\varepsilon(\{\varrho_a\})$ as follows: $g\in F$ is in $N_\varepsilon(\{\varrho_a\})$ if $g=e$ or if $g=g_1\ldots g_n$ is the reduced word representing g in F ([7]; hence $g_i\in G_{a_i}$, $a_i\neq a_{i+1}$) and there is some word $e=e_1\ldots e_n$ in F with e_i coming from the same factor as g_i, $1\leqslant i\leqslant n$, and

$$\sum_i \varrho_{a_i}(g_i, e_i)<\varepsilon.$$

Let the topology τ be the group topology on F for which the $N_\varepsilon(\{\varrho_a\})$ form an open neighborhood base at e. It is shown in [11] that τ makes F a locally invariant topological group and that sets open in τ are also open in the free product topology on F.

3. Local invariance.

PROPOSITION 3. *Let R_1 and R_2 be two copies of the additive group of reals with the usual topology. Then $F=R_1*R_2$ is not locally invariant.*

Proof. First, we introduce a group operation in the Cartesian product $G=R_1\times R_2$ such that G is a topological group but not locally invariant. Observation of Exercise 0 (d) of [5], p. 210, shows that

$$(a,b)\circ(c,d)=(a+c,\, b+e^a d)$$

is such an operation; $R_1\times R_2$ has the usual topology. Now, the natural injections $R_1\to G$ and $R_2\to G$ extend to a continuous homomorphism

$f: F \to G$ which is open and onto. Now, if F were locally invariant, G would be also; hence F is not locally invariant.

It is remarked in [1] that it is possible quite generally to find a semidirect product of two locally invariant groups which fails to be locally invariant. Clearly, the free product of any two such groups will also fail to be locally invariant. Since local invariance is inherited by subgroups, the free product of more than two factors fails to be locally invariant whenever the product of two of the factors does.

Since the topology τ, introduced in Section 2, makes F locally invariant, we have

COROLLARY 4. *The free product topology on* $F = R_1 * R_2$ *is strictly finer than* τ.

It would be nice to know if this is universal. In particular, is the free product of two circles $S^1 * S^1$ locally invariant? Since $S^1 \times S^1$ is compact, a different sort of argument would be needed.

If we let $F = * R_a$, $a \in A$, be a free product of connected, locally compact, locally invariant, Hausdorff groups such that F is not locally invariant, we see, by Theorem A, that each R_a is M.A.P., and thus F is M.A.P. and connected but not locally invariant, and hence not locally compact. This complements nicely the example given by Murakami [10] of a group which is M.A.P. and locally compact but neither connected nor locally invariant. In any such case we also obtain a corollary like the following

COROLLARY 5. *The continuous isomorphism of the M.A.P. group* $F = R_1 * R_2$ *into its Bohr compactification is not a homeomorphism onto a subgroup.*

Proof. If it were, $R_1 * R_2$, as a subgroup of a compact group, would be locally invariant.

Again, such a result would be of interest for $S^1 * S^1$.

4. Compactness and local compactness. Hulanicki [4] observed that a free product of compact Hausdorff groups is M.A.P., but left open the question whether it can be compact. Proposition 2 suggests that a free product of non-compact factors is non-compact; we see now that even a free product of compact factors is pathological. The proof of Theorem 6 was arrived at in a conversation with D. B. Coleman; it settles the problem of compactness and gives an indication of the solution for local compactness.

THEOREM 6. *Let* F *be the free product of Hausdorff topological groups* G_a, $a \in A$.

(1) *If at least two* G_a *are not* $\{e\}$, F *is not compact.*

(2) *If* F *is locally invariant and at least two* G_a *are not discrete,* F *is not locally compact.*

Proof. In case (1), suppose F is compact. Hence F and its subgroups G_a are locally invariant. Set $N = F$.

In case (2), suppose F is locally compact; then the identity $e \in F$ has a neighborhood N which is both compact and invariant. The G_a are locally invariant, since F is.

The following argument will show, in either case, that N is not compact. Let G_1 and G_2 be two non-trivial factors, and choose $g_i \in G_i \cap N \setminus \{e\}$, $i = 1, 2$. Write

$$h_n = (g_1 g_2)^n g_1 (g_1 g_2)^{-n};$$

then h_n is a reduced word of length $4n+1$, and $h_n \in N$ for all n. Since N is supposedly compact, some subnet $\{h_{n_\delta}\}_{\delta \in \Delta}$ converges to some $r \in N$, say $r = r_1 \ldots r_k$, and $r^{-1} h_{n_\delta} \to r^{-1} r = e$. The words $r^{-1} h_{n_\delta}$ "involve" altogether (in a reduced form) only finitely many letters (perhaps $r_k^{-1}, \ldots, r_1^{-1}$, $g_1^{\pm 1}$, $g_2^{\pm 1}$; fewer or others if, e.g., $r_1^{-1} \in G_1$). Now choose an invariant pseudometric ϱ_a on each G_a, $a \in A$, such that, for each "involved" letter (for concreteness, $r_i^{-1} \in G_{a_i}$), $\varrho_{a_i}(r_i^{-1}, e) \geqslant 1$.

In view of the remarks on the topology τ in Section 2, $r^{-1} h_{n_\delta} \to e$ implies that $r^{-1} h_{n_\delta}$ is eventually in $N_{1/2}(\{\varrho_a\})$. Thus, if $g_1 \ldots g_m$ is the reduced form of an appropriate $r^{-1} h_{n_\delta}$, there is some appropriate word $e = e_1 \ldots e_n$ with

$$\sum_i \varrho_{a_i}(g_i, e_i) < \frac{1}{2}.$$

But since $e_i \in G_{a_i}$, $a_i \neq a_{i+1}$, the word problem for free groups [7] guarantees some $e_i = e$. Then

$$\sum_i \varrho_{a_i}(g_i, e_i) \geqslant \min\{\varrho_{a_i}(g_i, e)\} \geqslant 1,$$

a contradiction, completing the proof.

We note that the non-triviality requirements are necessary. The free product of two discrete groups of order greater than one is an infinite discrete group, whence locally compact (although, of course, not compact). On the other hand, it seems highly probable that (2) holds without the local invariance condition; this can await a better description of the topology of F in the general case. The following corollaries settle local compactness for such obvious cases as $R * R$ and $S^1 * S^1$.

COROLLARY 7. *If the factors of F are non-trivial, connected, and M.A.P., then F is not locally compact.*

Proof. If F were locally compact, it would be a connected, locally compact, M.A.P. group, and thus, by Theorem A, locally invariant; so Theorem 6 (2) would apply.

COROLLARY 8. *If the factors of F are non-trivial, connected, locally invariant, and Hausdorff, then F is not locally compact.*

Proof. If F were locally compact, the factors would be closed (since F is Hausdorff by [11]), and thus locally compact. Then, by Theorem A, the factors would be M.A.P., and Corollary 7 would apply.

These results extend, by [6] and [10], to at least some non-connected groups.

We remark, briefly, on an interaction with Theorem A that gives less general results than those above, but seems interesting nonetheless. Suppose F to be connected and either M.A.P. or locally invariant. If F were locally compact, Theorem A would imply that F is the direct product of a compact group and a vector group. However, F is algebraically a free product, and thus cannot be written as a non-trivial direct product [7]; and since F is non-commutative, and thus not a vector group, it would have to be compact. Thus Theorem 6 could be proven more simply if F were assumed connected.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KENTUCKY
LEXINGTON, KENTUCKY

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