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THE FREE TOPOLOGICAL GROUP OVER THE RATIONALS

Temple H. FAY

Department of Mathematics, New Mexico State University, Las Cruces, NM 88003, USA

Edward T. ORDMAN

Department of Mathematics, New England College, Henniker, NH 03242, USA

Barbara V. SMITH THOMAS

Department of Mathematical Sciences, Memphis State University, Memphis, TN 38152, USA

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In this paper we investigate the topological structure of the Graev free topological group over the rationals. We show that this free group fails to be a k-space and fails to carry the weak topology generated by its subspaces of words of length less than or equal to n. As tools in this investigation we establish some properties of net convergence in free groups and also some properties of certain canonical maps which are closely related to the topological structure of free groups.

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0. Introduction

Free topological groups were first investigated in the 1940's by Graev [3], Kakutani [7], Markov [8], Samuel [13], and others; they have recently attracted considerable attention [3, 4, 6, 11, 12, 14] arising at least in part from the current interest in those "universal" properties enjoyed by all left adjoint functors. These universal properties provide very elegant proofs of some of the more elementary properties of free topological groups, and lead one to hope that the more subtle problems of the structure of the topology of free topological groups might yield to these same methods of attack. In fact, Ordman, in [12], has described completely the topological structure of the free topological group over a k_ω-space using the universal properties of "free" functors and k-coreflections. He showed that the free group over a k_ω-space is the weak union of the subsets consisting of words of length $\leqslant n$, and that each of these subsets carries the quotient topology induced by the canonical map

$$i_n : (X \cup_e X^{-1})^n \to F_G(X)_n,$$

which sends n-tuples

$$(x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1})$$

to n-letter words

$$x_1^{\pm 1} \cdot x_2^{\pm 1} \cdot \cdots \cdot x_n^{\pm 1}.$$

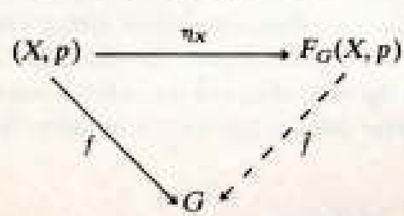
It remained an open question whether or not these results were true for more general topological spaces. In this paper we show that even for as nice a space as $\mathbb{Q}$ both of these results are false, and further that not only does the free group over $\mathbb{Q}$ fail to be a k-space, but that even the closed subset consisting of words of length ≤ 3 fails to be a k-space.

We choose to work with free topological groups of the Graev (pointed) type. In Section 1 we use the "universal" properties mentioned above to prove elegantly a number of preliminary results that are in print for Markov free topological groups and/or are part of the folklore for Graev groups. In Sections 2 and 3 we develop the machinery needed to prove the assertions about $\mathbb{Q}$ stated above. In section 2 we also use the properties of the maps i_n to discuss the structure of neighborhoods and compact sets in $F_G(X)_n$ the set of words of length not exceeding n. Section 4 contains the proofs of our main theorems.

1. Preliminaries

Our main references for this section are "Applications of the Stone-Cech compactification to free topological groups" by Hardy, Morris, and Thompson [5] and "Free topological groups" by Thomas [14]. Both of these papers discuss a slightly different notion of free topological groups than we are considering (the Markov rather than the Graev definition), but the proofs are sufficiently similar that we will only refer the reader to the corresponding result in one of these two papers.

1.1. Definition. The *Graev free topological group* over a pointed topological space (X, p) consists of a topological group $F_G(X, p)$ and a continuous function $\eta_X : (X, p) \to F_G(X, p)$, with $\eta_X(p) = e$, and with the property that any continuous function f from X to a topological group G, such that $f(p) = e_G$ "lifts" to a unique continuous group homomorphism $\hat{f} : F_G(X, p) \to G$, so that



commutes. Put more succinctly F_G is the left adjoint to the forgetful functor from the category of topological groups to the category of pointed topological spaces.

The existence of the functor F_G can be shown directly as in [14, Sections 1.2, 1.3 and 1.4]; by noting that $F_G(X,p)=F_M(X)/\langle p\rangle$, where $F_M(X)$ is the Markov free group over X and $\langle p\rangle$ is the normal subgroup generated by the singleton word p, or by using Wyler's notion of a TOP category [15,16].

We can describe $F_G(X,p)$ as follows: The underlying group of $F_G(X,p)$ is the usual free group on the set $X\setminus\{p\}$. The function $\eta_X:(X,p)\to F_G(X,p)$ is "insertion of generators" with $\eta_X(p)=e$. The topology of $F_G(X,p)$ can be described either as

(a) the finest topology compatible with the group structure such that η_X is continuous, or

(b) the weak topology induced by all group homomorphisms $\hat{f}:F_G(X,p)\to G$, where G is a topological group and $\hat{f}\circ\eta_X$ is continuous.

In fact, $F_G(X,p)$ is independent (up to homeomorphic isomorphism) of the choice of basepoint p in X, [3]. We will thus denote the Graev free group over a topological space by $F_G(X)$. Also, because the basepoint of X is the identity of $F_G(X)$ we will henceforth refer to it as e (or e_X) rather than p.

1.2. Theorem. *A topological space X is functionally Hausdorff if and only if $F_G(X)$ is Hausdorff.*

Proof. Recall that a topological space X is called functionally Hausdorff if any two points of X can be separated by a continuous real-valued function. This is equivalent to requiring that X can be mapped into a Tychonoff space by a one-to-one continuous function. If $F_G(X)$ is Hausdorff η_X is such a function. On the other hand, if X is functionally Hausdorff let $f:X\to Y$ be one-to-one and continuous with Y Tychonoff; then so also is $f:X\to\beta Y$. Let $e_{\beta Y}=f(e_X)$; Ordman has shown in [12] that $F_G(\beta Y)$ is Hausdorff and we can now consider f to be a map from X to $F_G(\beta Y)$. This f lifts to a one-to-one continuous group homomorphism $\hat{f}:F_G(X)\to F_G(\beta Y)$ and it follows, since $\{e_X\}=\hat{f}^{\leftarrow}[\{e_{\beta Y}\}]$, that $F_G(X)$ is Hausdorff. (We reserve the notation $^{-1}$ for inverses in groups, using $f^{\leftarrow}[\ ]$ for the inverse image under a map.)

1.3. Proposition. *A topological space X is completely regular (without Hausdorff) if and only if $\eta_X:X\to F_G(X)$ is an embedding.*

Proof. See [14, Section 3.1] and note that the restriction $f(e)=0$ is really no restriction at all.

1.4. Theorem. *A topological space X is Tychonoff (completely regular and Hausdorff) if and only if $\eta_X:X\to F_G(X)$ is a closed embedding.*

Proof. See [14, Section 0.2] and [5, Proposition].

1.5. Theorem. *If X' is a closed subspace of a Tychonoff space X containing the base point, then the subgroup of $F_G(X)$ generated by X' is closed.*

Proof. Let i denote the inclusion of X into βX; then i lifts to a continuous, one-to-one, group homomorphism $\hat{i}: F_G(X) \to F_G(\beta X)$. Let $C = \mathrm{cl}_{\beta X}(X')$, then C is a compact subspace of βX containing e and thus by [12, Proposition 5.4] $\langle C \rangle$,

$$\begin{array}{ccc} \langle X' \rangle & \hookrightarrow & F_G(X) \\ {\scriptstyle \hat{i}|_{\langle X' \rangle}} \downarrow & & \downarrow {\scriptstyle \hat{i}} \\ \langle C \rangle & \hookrightarrow & F_G(\beta X) \end{array}$$

the subgroup generated by C is closed in $F_G(\beta X)$. Considering the commutative square we see that $\langle X' \rangle = \hat{i}^{\leftarrow}[\langle C \rangle]$ and thus $\langle X' \rangle$ is closed in $F_G(X)$.

1.6. Theorem. *Let X be a Tychonoff space, then $F_G(X)$ contains a closed copy of X^n for every n.*

Proof. The proof for the Graev free group is the same as for the Markov free group and may be found in [5, Theorem A]. The embedding is defined by $(x_1, x_2, \ldots, x_n) \mapsto x_1 \cdot x_2^2 \cdot \cdots \cdot x_n^{2n-1}$.

1.7. Proposition. *F_G preserves quotient maps; that is, if $f: X \to X'$ is a quotient map, then the induced homomorphism $\hat{f}: F_G(X) \to F_G(X')$ is a quotient map.*

Proof. See [14, Section 3.9]. Note that, in fact, $\hat{f}$ is open.

1.8. Remark. By substituting "abelian topological group" for each occurrence of "topological group" in Definition 1.1 one obtains the definition of the *Graev free abelian topological group* $Z_G(X, p)$. All the theorems of this section also hold for $Z_G(X, p)$, with essentially the same proofs.

2. Compact sets and relatively open sets in $F_G(X)_n$

We now wish to investigate more closely the topology of $F_G(X)$ and a family of maps which arises in this investigation. Let $F_G(X)_n$ denote the collection of words of $F_G(X)$ which have reduced length n. (We will consider e to have length 0.) Let X^{-1} denote a disjoint copy of X and $X \cup_e X^{-1}$ the pointed union of X with X^{-1}; then for each n there is a "natural" map

$$i_n : (X \cup_e X^{-1})^n \to F_G(X)_n$$

defined by

$$i_n(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_n^{\varepsilon_n}) = x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \cdots \cdot x_n^{\varepsilon_n}$$

(for each i, $\varepsilon_i = \pm 1$). The map i_n is just the restriction to $(X \cup_e X^{-1})^n$ of the n-ary derived multiplication $m_n : (F_G(X))^n \to F_G(X)$ and is therefore continuous. If X should happen to be compact, then i_n is closed and $F_G(X)_n$ carries the quotient topology determined by i_n. In fact, Ordman has shown i_n to be a quotient map under the slightly more general requirement that X be a k_ω-space. (X is a k_ω-space if there is a sequence of compact subsets X_n of X such that $X = \cup X_n$ and $F \subset X$ is closed if and only if each $F \cap X_n$ is closed; see [12, 2]). In Section 4 we show that i_n need not be quotient in general. This is in marked contrast to the situation for free k-groups where $i_n : k(X \cup_e X^{-1})^n \to F_K(X)$ is always quotient; compare [12].

From now on all spaces are assumed to be Tychonoff. The following two propositions are well known.

2.1. Proposition. *For each n, $F_G(X)_n$ is closed in $F_G(X)$.*

Proof sketch (see [5, Proposition for details]). Consider the commutative square

$$\begin{array}{ccc}
(X \cup_e X^{-1})^n & \xrightarrow{\quad i_n \quad} & F_G(X) \\
\big\downarrow & & \big\downarrow \hat{\imath} \\
(\beta X \cup_e \beta X^{-1})^n & \xrightarrow{\quad i_n \quad} & F_G(\beta X)
\end{array}$$

Clearly $F_G(\beta X)_n$ is compact and hence closed in $F_G(\beta X)$, and $F_G(X)_n = \hat{\imath}^{\leftarrow}[F_G(\beta X)_n]$.

2.2. Proposition. *Every compact subset of $F_G(X)$ is contained in some $F_G(X)_n$.*

Proof. Consider again the canonical map $\hat{\imath} : F_G(X) \to F_G(\beta X)$; if C is a compact subset of $F_G(X)$, then $\hat{\imath}(C)$ is a compact subset of $F_G(\beta X)$. Now $F_G(\beta X)$ is a k_ω-space with the $F_G(\beta X)_n$ being the required sequence of compact subsets [12]. It follows that $\hat{\imath}(C)$ is in some $F_G(\beta X)_n$ which means that C is in $F_G(X)_n$.

From this one concludes that if X is not discrete $F_G(X)$ is not locally compact since any open subset of $F_G(X)$ must contain words of arbitrary length. (Let x, y be distinct points of X with x not isolated, and x_δ a net in $X \setminus \{x\}$ converging to x. Then $x_\delta \cdot x^{-1} \to e$ in $F_G(X)$, so $y^n x_\delta x^{-1} y^{-n} \to e$ and every neighborhood of e contains a word of length $2n+2$. So, therefore, does every open set in $F_G(X)$. In fact, local compactness may fail in $F_G(X)_n$ if X is not compact, as we shall show in Example 2.10 below.

2.3. Proposition. *If X is a metrizable k_ω-space, then every compact subset of $F_G(X)$ is metrizable.*

Proof. (For properties of k_ω-spaces see [2] or [12].) Write $X = \cup_k X_k$; then $(X \cup_e X^{-1})^n = \cup_k (X_k \cup_e X_k^{-1})^n$ is a k_ω-decomposition, and $F_G(X) = \cup_n i_n[(X_n \cup_e X_n^{-1})^n]$ is also. Hence any compact subset of $F_G(X)$ is actually contained in some $i_n[(X_n \cup_e X_n^{-1})^n]$ and is therefore metrizable.

Implicit in the proceeding is

2.4. Proposition. *If X is a k_ω-space the maps $i_n : (X \cup_e X^{-1})^n \to F_G(X)_n$ are compact covering.*

Proof. If X is a k_ω-space we know that $F_G(X)_n$ carries the quotient topology induced by i_n and that the sets $i_n[(X_k \cup_e X_n^{-1})^n]$ are a k_ω-decomposition of $F_G(X)_n$. Thus if C is a compact subset of $F_G(X)_n$ it is actually contained in some $i_n[(X_k \cup_e X_k^{-1})^n]$. It follows that C is the image of $i_n^{\leftarrow}[C] \cap (X_k \cup_e X_k^{-1})^n$, which is compact.

The importance of Proposition 2.4 lies in the fact that for X a k_ω-space it allows us to characterize the compact subsets of $F_G(X)$ as precisely the images $i_n[C]$ of compact subsets of the $(X \cup_e X^{-1})^n$. Since for k_ω-spaces X, $F_G(X)$ is a k-space this, in effect, determines the topology.

Let $F_K(X)$ denote the "free group" analogous to $F_G(X)$ in the category of k-groups [12]. The result about $F_K(X)$ which suggested Proposition 2.4 is as follows: Let X be a k-space. Then each $F_K(X)_n$ carries the quotient topology induced by i_n where the domain carries the k-coreflection of the product topology; and $F_K(X)$ is the weak union of the subsets $F_K(X)_n$. It is surprising that to test whether or not a subset of $F_K(X)_n$ is closed it suffices to check it against only the sets $i_n[(C \cup_e C^{-1})^n]$, for C any compact subset of X, rather than against all compact subsets of $F_K(X)_n$, and thus to check if a subset of $F_K(X)$ is closed it suffices to check it against all possible sets $i_n[(C \cup_e C^{-1})^n]$. All this seems to suggest some kind of compact-covering property for i_n; however, i_n does not have to be compact covering, either for $F_G(X)$ or for $F_K(X)$, even if X is locally compact, as the following example shows.

2.5. Example. Let ω_1 denote the countable ordinals with the order topology; let A denote the limit ordinals in ω_1. The quotient ω_1/A is compact, but no compact subset of ω_1 has ω_1/A as its image, and thus $q : \omega_1 \to \omega_1/A$ is not compact covering [10]. Let X be ω_1 with 0 as basepoint.

In what follows we are deliberately vague about whether $i_n[(X \cup_e X^{-1})^n]$ carries the subspace topology from $F_G(X)$ or $F_K(X)$ since the argument is valid in both cases. Now, $\omega_1 \cup_e \omega_1^{-1}$ can be considered to be $(-\omega_1, \omega_1)$ with the order topology, and so every compact subset of $(-\omega_1, \omega_1)^2 = (X \cup_e X^{-1})^2$ is contained in a closed square $[-\beta, \beta] \times [-\beta, \beta]$. Consider

$$Y = \{(\alpha^{-1}, \alpha+2) \mid \alpha \notin A\} \cup \{(\alpha^{-1}, \alpha) \mid \alpha \in A\} \subset (-\omega_1, \omega_1)^2.$$

The map $h : \omega_1 \to Y$ defined by

$$h(\alpha) = \begin{cases} (\alpha^{-1}, \alpha+2 & \alpha \notin A, \\ (\alpha^{-1}, \alpha) & \alpha \in A, \end{cases}$$

is clearly continuous, and therefore the composition

$$\omega_1 \xrightarrow{h} Y \hookrightarrow (-\omega_1, \omega_1)^2 \xrightarrow{i_2} i_2[(-\omega_1, \omega_1)^2]$$

is continuous also. Since every point of A is mapped to the identity this composition factors through ω_1/A, whence $i_2(Y)$ is compact. However no compact $C \subset [-\beta, \beta] \times [-\beta, \beta]$ can have $i_2(Y)$ as its image.

We now turn to developing an internal description of the topology $F_G(X)_n$ carries if i_n is a quotient map. It is based on the following proposition which was first proved by C. Joiner in [6].

2.6. Proposition. *Let $w = x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \cdots \cdot x_n^{\varepsilon_n}$ be a reduced word of length n in $F_G(X)_n$. Then the sets of the form $V_1^{\varepsilon_1} V_2^{\varepsilon_2} \cdots V_n^{\varepsilon_n} = \{y_1^{\varepsilon_1} \cdot y_2^{\varepsilon_2} \cdot \cdots \cdot y_n^{\varepsilon_n} \mid$ for each i, $y_i \in V_i\}$, where V_i is a neighborhood of x_i in X, form a neighborhood base for w in $F_G(X)_n$.*

It is crucial in this proposition that w has a unique representation in $F_G(X)_n$. If w were a word with reduced length less than n, then any open set containing w would, by the continuity of the multiplication of $F_G(X)$, have to contain such a $V_1^{\varepsilon_1} V_2^{\varepsilon_2} \cdots V_n^{\varepsilon_n}$ around *every* n-letter representation of w. We are led to

2.7. Definition. Call a subset S of $F_G(X)_n$ *J-open* if for every $w \in S$ and every n-letter representation $x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \cdots \cdot x_n^{\varepsilon_n}$ of w, S contains a set of the form $V_1^{\varepsilon_1} V_2^{\varepsilon_2} \cdots V_n^{\varepsilon_n}$, where each V_i is a neighborhood of x_i in $X \cup_e X^{-1}$.

In Lemma 2.9 below we show that the collection of J-open subsets of $F_G(X)_n$ coincides with the quotient topology induced by i_n, sparing us the necessity of verifying directly that the J-open sets form a topology.

2.8. Proposition. *Every open set in $F_G(X)_n$ is J-open.*

Proof. Immediate from the continuity of $i_n : (X \cup_e X^{-1})^n \to F_G(X)_n$ and the rectangular nature of open sets in $(X \cup_e X^{-1})^n$.

In Section 4 we shall have occasion to use nets which are eventually (frequently) in every J-open set containing a word $w \in F_G(X)_n$. Such nets will be called *J-convergent* (*J-clustering*), and by Proposition 2.8 they do, in fact, converge (cluster) in $F_G(X)_n$.

2.9. Lemma. *Every J-open subset of $F_G(X)_n$ is open if and only if i_n is a quotient map.*

Proof. First suppose that i_n is quotient. Let $S \subset F_G(X)_n$ be J-open; we will show that $i_n^{\leftarrow}[S]$ is open in $(X \cup_e X^{-1})^n$. Let

$$(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_n^{\varepsilon_n}) \in i_n^{\leftarrow}[S];$$

then $x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \cdots \cdot x_n^{\varepsilon_n}$ is an n-letter representation of some word w in S. Since S is J-open there exist neighborhoods V_i of x_i in $X \cup_e X^{-1}$ such that $V_1^{\varepsilon_1} V_2^{\varepsilon_2} \cdots V_n^{\varepsilon_n} \subset S$. But then

$$V_1^{\varepsilon_1} \times V_2^{\varepsilon_2} \times \cdots \times V_n^{\varepsilon_n} \subset i_n^{\leftarrow}[S]$$

which is therefore open.

Conversely, suppose that every J-open set is open, that $S \subset F_G(X)_n$ and that $i_n^{\leftarrow}[S]$ is open. Let w be a word in S and let $x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \cdots, x_n^{\varepsilon_n}$ be an n-letter representation of w. Then

$$(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_n^{\varepsilon_n}) \in i_n^{\leftarrow}[S]$$

and since $i_n^{\leftarrow}[S]$ is open it contains a neighborhood $V_1^{\varepsilon_1} \times V_2^{\varepsilon_2} \times \cdots \times V_n^{\varepsilon_n}$ of $(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_n^{\varepsilon_n})$, whence S is J-open and therefore open.

Given Lemma 2.9, it is in fact possible to extend the definition of J-open sets to all of $F_G(X)$. Let F_n denote the set $F_G(X)_n$ with the quotient topology received from i_n. We have a commutative diagram

$$\begin{array}{ccc} (X \cup_e X^{-1})^n & \xrightarrow{\;i_n\;} & F_n \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ (X \cup_e X^{-1})^{n+1} & \xrightarrow{\;i_n\;} & F_{n+1} \end{array}$$

for each $n \geq 0$, where p is the map $p(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}) = (x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}, e)$ and q turns out to be a closed embedding (to see this, compare Lemma 6.8 of [9], which is stated there in the k-category but is valid here; the division of $i_{n+1}^{\leftarrow}[q(F_n)] \subset (X \cup_e X^{-1})^{n+1}$ into finitely many closed subsets is very similar to the argument in Proposition 3.2 of [12]).

Now since we can treat each F_n as a subset of F_{n+1}, we can define a topology, called the J-topology, on the set $F_G(X)$ by: a set U is open in the J-topology (J-open) if and only if $U \cap F_n$ is open in F_n for each n.

If X is a k_ω-space, the J-topology coincides with the free group topology on $F_G(X)$. However, in general the J-topology may be strictly finer than the group topology, as may be seen from the examples in Section 4.

Now J-convergent (J-clustering) may be defined to mean convergent (clustering) in the J-topology, and it remains true that J-convergent (J-clustering) nets converge (cluster) in the group topology on $F_G(X)$.

We now return to the question of the local compactness of $F_G(X)_n$, using the preceeding lemma and the metrizability result above (Proposition 2.3).

2.10. Example. $F_G(\mathbb{R})_2$ is not locally compact. We actually show that $e = 0$ does not have a countable base of neighborhoods in $F_G(\mathbb{R})_2$ and hence cannot have a compact neighborhood, since such a compact neighborhood would have to be metrizable. Now, $\mathbb{R}$ is a k_ω-space, so i_2 is a quotient map and the open sets in $F_G(\mathbb{R})_2$ are precisely the J-open sets. Note also that

$$D = \{(r^{-1}, r) \mid r \in \mathbb{R}\setminus\{0\}\} \cup \{(0, 0)\}$$

is a closed subset of $(\mathbb{R} \cup_0 \mathbb{R}^{-1})^2$ contained in $i_2^{\leftarrow}[e]$. A countable $F_G(\mathbb{R})_2$-neighborhood base at $e = 0$ would thus induce a countable neighborhood base for D and that is impossible.

3. Net convergence in $F_G(X)_n$

To analyse further the structure of $F_G(X)_n$ it will be useful to establish some properties of net convergence there. Suppose we have a net $(w_\alpha)_{\alpha \in \mathscr{A}}$, with each $w_\alpha \in F_G(X)_n$, which converges to a word w. The w has reduced length $\leq n$. If w has reduced length precisely n then $(w_\alpha)_{\alpha \in \mathscr{A}}$ converges pointwise by Joiner's Fundamental Lemma (our Proposition 2.6); however, if w has reduced length less then n the situation can be considerably more complicated. For, consider the following sequence in $F_G(\mathbb{R})_2$:

$$w_n = \begin{cases} \frac{1}{n} \cdot 2 & n \text{ even}, \\ 2 \cdot \frac{1}{n} & n \text{ odd}. \end{cases}$$

This sequence converges to the singleton word 2, but does not converge pointwise. Worse yet can happen as we shall see in Example 3.7.

We start our consideration of convergent nets by showing that a net which converges pointwise is convergent.

3.1. Lemma. *If*

$$(w_\alpha)_{\alpha\in\mathcal{A}} = (x_{1\alpha}^{\varepsilon_{1\alpha}} x_{2\alpha}^{\varepsilon_{2\alpha}} \cdots x_{k_\alpha\alpha}^{\varepsilon_{k_\alpha\alpha}})_{\alpha\in\mathcal{A}}$$

is a net in $F_G(X)_n$ *such that for each* i, $\varepsilon_{i\alpha}$ *is constant and* $x_{i\alpha}$ *converges to* x_i *in* X, *then* $(w_\alpha)_{\alpha\in\mathcal{A}}$ *converges to* $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$.

Proof. Multiplication is continuous and X is a subspace of $F_G(X)$.

3.2. Lemma. *If*

$$(w_\alpha)_{\alpha\in\mathcal{A}} = (x_{1\alpha}^{\varepsilon_1} x_{2\alpha}^{\varepsilon_2} \cdots x_{k_\alpha\alpha}^{\varepsilon_{k_\alpha}})_{\alpha\in\mathcal{A}}$$

net in $F_G(X)_m$, *then there is a subnet* $(w_\delta)_{\delta\in\mathcal{D}}$ *such that*

(i) *for all* δ, $k_\delta = k$ *(the words* w_δ *have fixed length* k*),*

(ii) *for each* i, $\varepsilon_{i\delta}$ *is constant,*

(iii) *for each* i, *either* $x_{i\delta}$ *converges in* X *or it fails even to cluster in* X.

Proof. Obvious, but note that the finite word length is crucial.

Note that condition (iii) above can be replaced (possibly changing the subnet) by

(iii') *for each* i, $x_{i\delta}$ *converges in* βX.

3.3. Definition. Call a net $(w_\alpha)_{\alpha\in\mathcal{A}}$ in $F_G(X)_n$ *β-pointwise convergent* if it satisfies conditions (i), (ii), and (iii') above.

3.4. Example. From Lemma 3.1 we see that β-pointwise convergence implies convergence in $F_G(\beta X)$; this convergence might be to a word in $F_G(\beta X)$ having a reduced form containing only letters from X. (For instance, let $y \in \beta X\setminus X$; the net $w_\alpha = x_{1\alpha} x_{2\alpha}^{-1}$ could converge β-pointwise to $y \cdot y^{-1} = e \in F_G(\beta X)$.) However, even in such a case, β-pointwise convergence of (w_α) does not necessarily imply convergence of (w_α) in $F_G(X)$. For, let $X = \mathbb{N}$, and let $y \in \beta\mathbb{N}\setminus\mathbb{N}$. Define two nets on $\mathbb{N}$ converging to y in $\beta\mathbb{N}$ by choosing two distinct points in $\mathbb{N}$ from each neighborhood F of the point y, call them x_{1F} and x_{2F}. Then $(x_{1F}^{+1} \cdot x_{2F}^{-1})_{F\in\mathcal{N}(y)}$ is a non-constant β-pointwise convergent net in $F_G(\mathbb{N})_2$, converging to the identity in $F_G(\beta\mathbb{N})$, which cannot converge in $F_G(\mathbb{N})$ since $F_G(\mathbb{N})$ is discrete.

3.5. Lemma. *Let* $(w_\alpha)_{\alpha\in\mathcal{A}}$ *be a net in* $F_G(X)_n$ *which converges to* w. *Then there exists a* β*-pointwise convergent subnet* $(w_\delta)_{\delta\in\mathcal{D}}$ *whose limit in* $F_G(\beta X)_n$ *reduces to* w.

Proof. Using Lemma 3.2 with condition (iii') produces the subnet $(w_\delta)_{\delta\in\mathcal{D}}$; denote its limit in $F_G(\beta X)$ by $\hat{w}$. Use the canonical $\hat{i}: F_G(X) \to F_G(\beta X)$ to see that $w_\delta = \hat{i}(w_\delta)$ converges to $\hat{i}(w) = w$. Thus $\hat{w}$ reduces to w.

3.6. Corollary. *If* X *is compact Hausdorff and if* $w_\alpha \to w$ *in* $F_G(X)$, *then there exists a subnet* $(w_\delta)_{\delta\in\mathcal{D}}$ *which converges pointwise to a word which reduces to* w.

3.7. Example. The "β-pointwise convergence" in Lemma 3.5 cannot be improved to "pointwise convergence" even for k_ω-spaces. Let $X = \mathbb{R}$, and consider the directed set $\mathbb{N}\times\mathbb{N}$ with the lexicographic order. We show below that the net $w_{n,j} = n^{-1}\cdot(n+1/j)$ clusters at $e=0$. Thus there is a subnet $(w_\alpha)_{\alpha\in\mathscr{A}}$ which converges to $e=0$, but no subnet of $(w_\alpha)_{\alpha\in\mathscr{A}}$ can converge pointwise to a representative of e in $F_G(\mathbb{R})_2$. To see that $(w_{n,j})$ clusters at e choose any $(n_0, j_0)\in\mathbb{N}\times\mathbb{N}$ and choose any open neighborhood U of e. Now U is J-open so for any $n > n_0$ there is an ε_n so that $(n-\varepsilon_n, n+\varepsilon_n)^{-1}\cdot(n-\varepsilon_n, n+\varepsilon_n)\subset U$. Pick j so that $1/j<\varepsilon_n$; then

$$w_{n,j} = n^{-1}\cdot(n+1/j)\in(n-\varepsilon_n, n+\varepsilon_n)^{-1}\cdot(n-\varepsilon_n, n+\varepsilon_n)\subset U,$$

and since $n > n_0$, $(n,j) > (n_0, j_0)$.

One does not always need to expand X "all the way" to βX to use the ideas of this section. For instance, we will later use:

3.8. Lemma. *Let $f: X\to Y$ be a continuous mapping of Tychonoff spaces, so that the induced homomorphism $\hat{f}: F_G(X)\to F_G(Y)$ is also continuous. Let $(w_\alpha)_{\alpha\in\mathscr{A}}$ be a net in $F_G(X)$ such that $\hat{f}(w_\alpha)_{\alpha\in\mathscr{A}}$ converges to a point v of $F_G(Y)\setminus\hat{f}(F_G(X))$. Then no subnet of $(w_\alpha)_{\alpha\in\mathscr{A}}$ converges in $F_G(X)$.*

Proof. If a subnet $(w_\delta)_{\delta\in\mathscr{D}}$ converged to $w\in F_G(X)$, then $\hat{f}(w_\delta)$ would converge to $\hat{f}(w)$ as well as to v, a contradiction.

4. Concerning $F_G(\mathbb{Q})$

We are now in a position to explore the topology of $F_G(\mathbb{Q})$. We show that the very nice description of the topology of $F_G(X)$ for k_ω-spaces (each i_n quotient and $F_G(X)$ having the weak union topology of the $F_G(X)_n$) fails on both counts for $F_G(\mathbb{Q})$. We first show that $F_G(\mathbb{Q})$ does not have the weak union topology of the $F_G(\mathbb{Q})_n$ by producing a net which clusters at $e=0$ in $F_G(\mathbb{Q})$ but whose intersection with each $F_G(\mathbb{Q})_n$ is closed.

4.1. Theorem. *$F_G(\mathbb{Q})$ does not have the weak union topology induced by its subspaces $F_G(\mathbb{Q})_n$.*

Proof. For each $n\in\mathbb{N}$ choose a decreasing sequence $(q_{n,j})_{j\in\mathbb{N}}$ in $\mathbb{Q}$ which converges in $\mathbb{R}$ to π/n and such that for all $n>1$, $\pi/n - 1 > q_{n,1}$. Give $\mathbb{N}\times\mathbb{N}$ the lexicographic order, then the net $(q_{n,j})_{(n,j)\in\mathbb{N}\times\mathbb{N}}$ converges to 0 in $\mathbb{Q}$, and hence to $e=0$ in $F_G(\mathbb{Q})$. Define a second net $t: \mathbb{N}\times\mathbb{N}\to F_G(\mathbb{Q})$ by

$$t_{n,j} = \frac{1}{n}\cdot\frac{1}{j}\cdot\frac{1}{j}\cdot\;\cdots\;\cdot\frac{1}{j},$$

where $1/j$ is repeated n times. Now, $(t_{n,j})$ clusters at $e=0$ for if U is an $F_G(\mathbb{Q})$-

neighborhood of 0 and if $(n_0, j_0) \in \mathbb{N} \times \mathbb{N}$ then $U \cap \mathbb{Q}$ is a $\mathbb{Q}$-neighborhood of 0, so there is an $n_1 > n_0$ such that $1/n_1 \in U \cap \mathbb{Q}$. Consider the sequence

$$(t_{n_1,j})_{j \in \mathbb{N}} = \left(\frac{1}{n_1} \cdot \frac{1}{j} \cdot \frac{1}{j} \cdot \cdots \cdot \frac{1}{j}\right)$$

in $F_G(\mathbb{Q})_{n_1+1}$. This sequence converges pointwise to the word

$$\frac{1}{n_1} \cdot 0 \cdot 0 \cdot \cdots \cdot 0 = \frac{1}{n_1},$$

and is thus eventually in $U \cap F_G(\mathbb{Q})_{n_1+1} \subset U$. It follows that the product net

$$(q_{n,j} \cdot t_{n,j}) = \left(q_{n,j} \cdot \frac{1}{n} \cdot \frac{1}{j} \cdot \frac{1}{j} \cdot \cdots \cdot \frac{1}{j}\right)$$

clusters at $0 \in F_G(\mathbb{Q})$. However, for all $k \geq 1$,

$$\{q_{n,j} \cdot t_{n,j}\} \cap F_G(\mathbb{Q})_{k+2} = \{q_{n,j} \cdot t_{n,j}\}_{(n,j) < (k+1,1)}$$

which is closed in $F_G(\mathbb{Q})_{k+2}$ (apply Lemma 3.8; the "irrational" points $\pi/n \cdot 1/n$ are not there).

Notice that this example already shows that $F_G(\mathbb{Q})$ does not have the same topology as $F_K(\mathbb{Q})$.

4.2. Corollary. *$F_G(\mathbb{Q})$ is not a k-space.*

Proof. The set $\{q_{n,j} \cdot t_{n,j} \mid (n, j) \in \mathbb{N} \times \mathbb{N}\}$ described above meets each $F_G(\mathbb{Q})_m$, and hence each compact subset of $F_G(\mathbb{Q})$, in a closed set; yet it is not closed, having $e = 0$ as a cluster point.

This corollary is interesting since $\mathbb{Q}$ is not only a k-space, but so are its finite powers; each $\mathbb{Q}^n$ is even metrizable. Previous examples showing that the free topological group over a k-space could fail to be a k-space did so by using Theorem 1.6 and finding a finite power X^n which was not k.

We turn now to showing that i_n need not be a quotient map for $n > 2$.

4.3. Theorem. *The map $i_3 : (\mathbb{Q} \cup_e \mathbb{Q}^{-1})^3 \to F_G(\mathbb{Q})_3$ is not quotient.*

Proof. Our strategy is to find a J-closed subset of $F_G(\mathbb{Q})_3$ which is not closed. To do this we find a J-convergent net and a J-clustering net whose product is J-closed; but the product must cluster in the group topology and thus cannot be closed there. The J-convergent (hence convergent) net is the net $(q_{n,j})$ of Theorem 4.1. The J-clustering net is the net $w_{n,j} = n^{-1} \cdot (n + (1/j))$ of Example 2.7; note that the argument that $(w_{n,j})$ clusters in $F_G(\mathbb{R})_2$ is really an argument that $(w_{n,j})$ J-clusters, and is valid word for word in $F_G(\mathbb{Q})_2$. The product of these two nets,

$(q_{n,j} \cdot n^{-1} \cdot (n+(1/j)))$, must cluster at $e=0$ by the continuity of the multiplication. However,

$$i_3^{\leftarrow}\left[\left\{q_{n,j} \cdot n^{-1} \cdot \left(n+\frac{1}{j}\right)\right\}\right]=\left\{\left(q_{n,j}, n^{-1}, n+\frac{1}{j}\right)\right\} \subset (\mathbb{Q} \cup_e \mathbb{Q}^{-1})^3$$

is closed. To see this, note that if (a, b, c) is a limit point of the set, we must have $b=n_0^{-1}$ for some n_0, and thus a must be a cluster point of $\{q_{n,j} \mid n=n_0\}$; the only such cluster point is π/n_0, which is not in $\mathbb{Q}$. Finally, since $i_3^{\leftarrow}[\{q_{n,j} \cdot n^{-1} \cdot (n+(1/j))\}]$ is closed, but $\{q_{n,j} \cdot n^{-1} \cdot (n+1/j)\}$ is not, i_3 is not a quotient map.

It is clear that i_p is not a quotient map for $p>3$; let the net be

$$\left(q_{n,j} \cdot n^{-1} \cdot \left(n+\frac{1}{j}\right) \cdot \frac{1}{n} \cdot \frac{1}{n} \cdot \cdots \cdot \frac{1}{n}\right),$$

adding as many $1/n$ terms as necessary.

4.4. Theorem. *$F_G(\mathbb{Q})_3$ is not a k-space.*

Proof. We again use the subset

$$F=\left\{q_{n,j} \cdot n^{-1} \cdot \left(n+\frac{1}{j}\right)\right\}_{(n,j)\in\mathbb{N}\times\mathbb{N}}$$

of Theorem 4.3. We have already established that this set is not closed in $F_G(\mathbb{Q})_3$ so it only remains to show that its intersection with any compact subset C of $F_G(\mathbb{Q})_3$ is closed. What we actually show is that if $F \cap C$ fails to be closed, then C cannot be compact. So let $w \in \mathrm{cl}(F \cap C) \setminus (F \cap C)$ and let $(w_\alpha)_{\alpha\in\mathcal{A}}$ be a net in $F \cap C$ which converges to w. Each w_α is of the form

$$q_{n(\alpha),j(\alpha)} \cdot n(\alpha)^{-1} \cdot \left(n(\alpha)+\frac{1}{j(\alpha)}\right),$$

and using Lemma 3.5 we may assume that $(w_\alpha)_{\alpha\in\mathcal{A}}$ β-pointwise converges to $x_1 \cdot x_2^{-1} \cdot x_3=w$, where x_1, x_2, and x_3 are in $\beta\mathbb{Q}$.

For some n_0 there are infinitely many j such that $(n(\alpha), j(\alpha))=(n_0, j)$. For suppose not, then for each n there are at most finitely many j such that $(n(\alpha), j(\alpha))=(n, j)$ and it follows that $\{n(\alpha)+(1/j(\alpha)) \mid \alpha \in \mathcal{A}\}$ and $\{n(\alpha) \mid \alpha \in \mathcal{A}\}$ are disjoint closed discrete subsets of $\mathbb{Q}$. Using the normality of $\mathbb{Q}$ we see that $x_3 \in \mathrm{cl}_{\beta\mathbb{Q}}\{n(\alpha)+(1/j(\alpha))\}$ and $x_2 \in \mathrm{cl}_{\beta\mathbb{Q}}\{n(\alpha)\}$ are distinct. Now x_3 must be in $\beta\mathbb{Q}\setminus\mathbb{Q}$, since if it is in $\mathbb{Q}$, w_α is eventually constant, which is impossible, and thus $n(\alpha)\to\infty$ which also implies that $x_2 \in \beta\mathbb{Q}\setminus\mathbb{Q}$. Hence $x_1 \cdot x_2^{-1} \cdot x_3$ cannot reduce to a word in $F_G(\mathbb{Q})_3$. Thus we have established that for some n_0 there are infinitely many j's.

Hence C contains infinitely many points $q_{n_0,j} \cdot n_0^{-1} \cdot (n_0+(1/j))$. Clearly this sequence converges in $F_G(\mathbb{R})$ to $\pi/n_0 \cdot n_0^{-1} \cdot n_0=\pi/n_0$, and so applying Lemma 3.8

we see that no subset of $(q_{n_0,j} \cdot n_0^{-1} \cdot (n_0+(1/j))$ can converge in $F_G(\mathbb{Q})$. Therefore C is not compact.

We have actually established by this theorem that *if* $n \geq 3$, *then* $F_G(\mathbb{Q})_n$ *is not a k-space* since it contains $F_G(\mathbb{Q})_3$ as a closed subspace.

We conclude with two questions suggested by the preceding results. First, every construction in this section depends heavily on $\mathbb{Q}$'s lack of local compactness, leading to:

4.5. Question. Is i_n always a quotient map if X is locally compact?

Second, we know that i_1 is a closed embedding for all Tychonoff spaces X and that i_3 is not a quotient map for $X = \mathbb{Q}$.

4.6. Question. Is $i_2 : (\mathbb{Q} \cup_e \mathbb{Q}^{-1})^2 \to F_G(\mathbb{Q})_2$ a quotient map? Is i_2 always a quotient map?

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