

## On Functions Defined by Iterations of Each Other<sup>1)</sup>

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### 1. Introduction

Let  $f$  and  $g$  be functions from and into the nonnegative integers, and let iteration of a function be denoted by superscripting (so that  $f^2(n) = ff(n) = f(f(n))$ ). We shall prove

**THEOREM.** *The pairs  $(f, g)$  of functions satisfying*

$$f^{m+1}(n) = g^{n+1}(m), \quad 0 \leq m, n < \infty \quad (1)$$

are precisely those given by

$$f(n) = g(n) = \begin{cases} n+1 & \text{for } n < k \\ a & \text{for } n = k \\ a_i & \text{for } n > k, n \equiv i \pmod{p} \end{cases} \quad (2)$$

where  $-1 \leq k \leq \infty$  and (if  $k < \infty$ )  $p, a, a_i$  are integers satisfying  $1 \leq p < \infty, 1 \leq i \leq p, a, a_i > k, a \equiv k+1 \pmod{p}$  and  $a_i \equiv i+1 \pmod{p}$ .

From this it follows that

**COROLLARY.** *The function  $f$  satisfies*

$$f(n) = f^{n+1}(0), \quad n \geq 0 \quad (3)$$

if and only if  $f$  satisfies (2).

*Proof of Corollary.* If  $f$  is as in (2), the pair  $(f, f)$  satisfies (1) by the theorem and (3) is the special case  $m=0$ . Conversely, if  $f$  satisfies (3), we can define  $g=f$  to give  $f^{m+1}(n) = f^m f(n) = f^m f^{n+1}(0) = g^m g^{n+1}(0) = g^n g(m) = g^{n+1}(m)$ . Then, since  $(f, g)$  satisfies (1),  $f$  must be as in (2).

The problem and many of the proofs may be pictured in terms of an array  $\{a_{m,n}\}$  of integers with  $a_{m,n} = f^{m+1}(n) = g^{n+1}(m)$ . The crucial fact in this interpretation is that a number such as  $f^3(0) = f^2 f(0) = ff^2(0)$  appears in each of the positions  $a_{2,0}, a_{1,f(0)}$ , and  $a_{0,f^2(0)}$ .

In figures 1, 2, and 3,  $f(n)$  is directly to the right of, and  $g(n)$  directly above, each occurrence of  $n$ .

<sup>1)</sup> This work was done at the National Bureau of Standards, Washington, D.C.

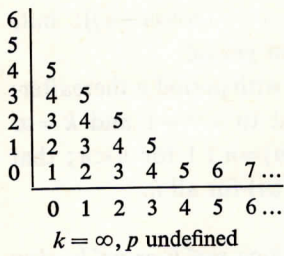


Figure 1

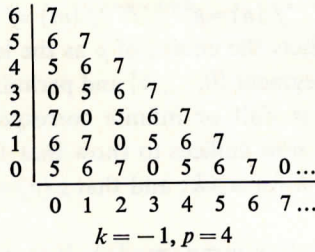


Figure 2

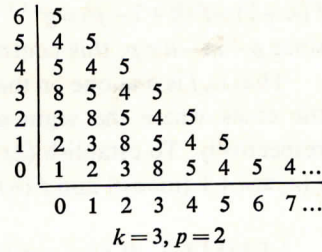


Figure 3

**2. Proof that (2) implies (1)**

Let  $a_{m,n}$  be given by

$$a_{m,n} = \begin{cases} m + n + 1 & \text{for } m + n < k \\ a & \text{for } m + n = k \\ a_i & \text{for } m + n > k, m + n \equiv i \pmod{p} \end{cases} \tag{4}$$

where the parameters are defined since  $f$  and  $g$  satisfy (2). We shall show that  $f^{m+1}(n) = a_{m,n}$ ;  $g = f$  by hypothesis and clearly  $a_{m,n} = a_{n,m}$ , so it will follow that  $f^{m+1}(n) = a_{n,m} = f^{n+1}(m) = g^{n+1}(m)$  completing the proof.

Formulae (2) and (4) make clear that  $a_{0,n} = f(n)$ ; we shall prove  $a_{m,n} = f^{m+1}(n)$  by induction on  $m$ , by showing that for all  $m, n, f(a_{m,n}) = a_{m+1,n}$ . There are four cases, for  $m+n$  respectively less than  $k-1$ , equal to  $k-1$ , equal to  $k$ , and greater than  $k$ . All are elementary; we illustrate with  $m+n = k$ . In this case  $a_{m,n} = a$ , so  $f(a_{m,n}) = f(a) = a_i$  where  $i \equiv a \equiv k+1 \pmod{p}$ . However,  $a_{m+1,n} = a_i$  where  $i \equiv (m+1) + n = k+1 \pmod{p}$ , so  $f(a_{m,n}) = a_{m+1,n}$ .

**3. Proof that (1) implies (2)**

LEMMA 1.  $g^m f^r(n) = f^r(m+n)$

*Proof.* Using (1) twice,  $g^m f^r(n) = g^m g^{n+1}(r-1) = g^{m+n+1}(r-1) = f^r(m+n)$ .

LEMMA 2. *There are integers  $k, -1 \leq k \leq \infty$ , and (if  $k < \infty$ )  $p, 1 \leq p < \infty$ , such that the values  $f(0), \dots, f(k)$  are assumed by  $f$  only at the points  $0, \dots, k$  respectively, and such that for all  $r > 0, f(k+r) = f(k+r+p)$ , while the values  $f(k+1), \dots, f(k+p)$  are distinct.*

*Proof.* If  $f$  is one-one, then  $k = \infty$ . If not, let  $k$  be the least integer such that  $f(k+1)$  is the image of more than one point, and let  $p > 0$  be the least integer for which  $f(k+1+p) = f(k+1)$ . Then, using Lemma 1 twice,  $f(k+r) = g^{r-1} f(k+1) = g^{r-1} f(k+1+p) = f(k+r+p)$  for all  $r > 0$ . To show that  $f(k+1), \dots, f(k+p)$  are distinct, suppose that  $k+1 \leq m < n \leq k+p$  and  $f(m) = f(n)$ . Again, using Lemma 1 twice,

$f(k+1) = f(k+1+p) = g^{k+1+p-n} f(n) = g^{k+1+p-n} f(m) = f(k+1+(p+m-n))$ ; but, since  $p+m-n < p$ , this contradicts the choice of  $p$  as the least period.

That is,  $f$  is one-one on the segment  $\{0, \dots, k\}$  and periodic with period  $p$  thereafter; the cases where the segment is null or infinite correspond to  $k = -1$  and  $k = \infty$  respectively. To establish (2) it now suffices to show that  $f(n) = n+1$  for  $n < k$ ; that  $f(n) \equiv n+1 \pmod{p}$  and  $f(n) > k$  for  $n \geq k$ ; and that  $g(n) = f(n)$  for all  $n$ .

**LEMMA 3.** *If  $f(m) = f(n)$ , then  $m \equiv n \pmod{p}$ . If in addition  $m \leq k$  or  $n \leq k$ , then  $m = n$ .*

*Proof.* If  $m$  and  $n$  are each greater than  $k$ , they must differ by a multiple of the period since the values in the period are distinct. If  $m \leq k$ ,  $f(m)$  is a value assumed only at the point  $m$ , so  $n = m$ .

**LEMMA 4.**  $f^{n+1}(0) = f(nf(0))$ .

*Proof.* This is clearly true for  $n = 0$ ; suppose it is true for some  $m$ . Then by Lemma 1,

$$\begin{aligned} f^{(m+1)+1}(0) &= f^{m+1}(f(0)) = g^{f(0)} f^{m+1}(0) \\ &= g^{f(0)} f(mf(0)) = f(mf(0) + f(0)) = f((m+1)f(0)), \end{aligned}$$

completing the induction.

**PROPOSITION 1.** *If  $k \neq \infty$  then  $f(n) \equiv n+1 \pmod{p}$ .*

*Proof.* By Lemma 1, formula (1), and Lemma 4,  $f(n+1) = gf(n) = f^{f(n)+1}(0) = f(f(n) \cdot f(0))$ ; by Lemma 3,

$$n+1 \equiv f(n) \cdot f(0) \pmod{p}. \quad (5)$$

Lemma 1 also yields  $ff(n) = g^n ff(0) = f(f(0) + n)$ , so by Lemma 3,

$$f(n) \equiv f(0) + n \pmod{p}. \quad (6)$$

Combining these,  $f(n) - f(0) + 1 \equiv n+1 \equiv f(n) \cdot f(0) \pmod{p}$ ; rearranging,  $f(n) + 1 \equiv f(0) \cdot (f(n) + 1) \pmod{p}$ . By (6), there is an  $n$  such that  $f(n) + 1$  is relatively prime to  $p$ ; thus we may divide by  $f(n) + 1$  to get  $f(0) \equiv 1 \pmod{p}$  and  $f(n) \equiv n+1 \pmod{p}$  for all  $n$ .

**PROPOSITION 2.** *If  $n < k$ , then  $f(n) = n+1$ .*

*Proof.* Where  $n < k$  (including the case  $k = \infty$ ) the proof of Proposition 1 holds with congruences replaced by equality. (5) holds since  $n < k$  implies  $n+1 \leq k$ ; then (5) implies  $f(n) \leq k$  so (6) holds, by Lemma 3. In this case (5) implies  $f(0) = 1$  immediately, since  $f(0)$  is nonnegative by hypothesis.

**PROPOSITION 3.**  $f(n) > k$  for  $n \geq k$ .

*Proof.* If  $k \geq 0$ ,  $f$  never assumes the value 0. If it did,  $f(r) = 0$  would imply  $f(r+1) = gf(r) = g(0) = f(0)$  would be the image of more than one point, and  $k$  would equal  $-1$ . If  $k > 0$ ,  $f$  assumes the values  $1, 2, \dots, k$  at  $0, 1, \dots, k-1$  and only there. Thus at all other points,  $f(n) > k$ .

**PROPOSITION 4.**  $g(n) = f(n)$  for all  $n$ .

*Proof.* Since  $(f, g)$  satisfies (1),  $g(n) = f^{n+1}(0)$ . But we have shown that  $f$  satisfies (2), so  $(f, f)$  satisfies (1) and thus  $f^{n+1}(0) = f(n)$ . Thus  $g(n) = f(n)$  and the proof that (1) implies (2) is complete.

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