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CONVERGENCE AND ABSTRACT SPACES IN FUNCTIONAL ANALYSIS[1]

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0. INTRODUCTION.

Since there are many notions of "convergence" for sequences of functions, it is tempting to use "convergence" rather than "open set" as a primitive concept in many problems of topology and functional analysis. The last five to ten years have seen substantial development of the theory of "limit spaces", spaces in which only a notion of convergence is given. This paper is an attempt to collect a few of the more important and interesting applications of the use of convergence as a primitive notion.

Earlier writers have used both nets and filters, and conditions often may be expressed more easily in terms of one than the other. Therefore, an initial section has been devoted to a condensed development of both nets and filters and the relationship between them. While later in the paper "convergence" is thought of in terms of "convergence of nets", filters are used freely in stating conditions and giving examples. The reader should be warned that terminology has been adapted somewhat, e.g. $\mathcal{L}$ space here denotes a space with convergent *nets* rather that the usual convergent *sequences*.

In the second section, several of the definitions that have been given for limit spaces are stated and compared, including initial standards that a limit space be a topological space, i.e., that there be a topology whose induced convergence is the one originally given. It is of interest that there are in fact important notions of convergence which are not induced by any topology.

While the second section discusses topologies only briefly, the third gives a brief introduction to general topology from a limit space point of view. An attempt has been made to weaken hypotheses of some standard theorems, although usually not as far as possible (in many cases this author is uncertain how weak the hypotheses may be). A decision has been made to indicate beginnings in many areas of topology, rather than to develop one or a few at greater length.

The remainder of the paper is devoted to the more complex structures of functional analysis, with an eye especially on function spaces. In particular, a notion of convergence on a set may correspond to the notion of convergence induced by a linear topological space, metric space or normed space; a few tests

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are given for examining such structures. The fourth section is concerned with notions of linearity, metrizability, and boundedness; the fifth with convexity and normability (the fourth and fifth sections will be in part 2).

A striking motivation for some of the topics to be discussed will be seen in the following theorem, which follows fairly easily from later parts of the paper:

0.1. THEOREM. *Consider the space of bounded functions from the reals into the reals, omitting nonmeasurable functions in* b. *Then* (a) *uniform convergence determines a normed linear space* (5.17), (b) *convergence in measure determines a pseudo-metrizable but non-normable linear topological space (metrizable if functions equal, i.e., are identified)* (4.16, 5.9, 5.11, A), (c) *pointwise convergence determines a non-metrizable linear topological space* (4.5, 4.10) *and* (d) *convergence almost everywhere is not equivalent to the convergence determined by any topology on the space* (2.18).

1. NETS AND FILTERS.

1.1. NETS. A *directed set* $(D,\geq)$ is a nonvoid set partially ordered by transitive and reflexive relation $\geq$ such that for every $a, b \in D$ there is a $c \in D$ satisfying $c \geq a$, $c \geq b$. A *net* $\{f_d,D,\geq\}$ in a set X is a function f(d), also denoted $f_d:D \to X$; it is often denoted $\{f_d, d \in D\}$ or simply f. A subset of D which includes some $b \geq a$ for each a in D is *cofinal*; f_d is *frequently* in A if $f_d \in A$ for all a in a cofinal subset of D, and *eventually* in A if $f_a \in A$ for all a following some fixed $b \in D$. The notation f_n will be reserved for *sequences* (nets whose domain is the natural numbers).

A net $\{T_e, e \in E,>\}$ is a *subnet* of $\{S_d, d \in D,\geq\}$ if there is a net $\{R_e, e \in E,>\}$ in D which eventually follows each point of D (i.e., for $a \in D$ there is $b \in E$ such that $e > b$ implies $R_e \geq a$) such that R satisfies the condition $T = S\cdot R$ (i.e., $T_e = T(e) = S(R(e)) = S(R_e)$). The domain of T thus need not have any apparent relationship to that of S; however, the range of T will be a subset of the range of S. It will occasionally be convenient to say that R *subordinates* T to S.

Suppose $(D_i,>_i)$ is a directed set for each i in an index set I. The *product directed set* $\Pi_i D_i$ is ordered componentwise: we write $x(\Pi_i>_i)y$, or simply $x > y$, if for each component $x_i >_i y_i$. This is clearly a partial order, and gives a directed set since an upper bound to two elements may be found componentwise. In particular, we write $(D,\geq)x(E,\leq) = (DxE,\geq\leq)$. If for each i $\{S_i,D_i,\geq_i\}$ is a net in X_i, the product net $\{\Pi S_i,\Pi D_i,\Pi\geq_i\}$ is a net in ΠX_i. We also make the following notational convention: if $\{f,D\}$ and $\{g,E\}$ are nets in X and p(a,b) is a function of domain X, then $p(f_d,g_e)$ will denote $\{p(f_d,g_e),(d,e) \in D \times E\}$, even if $D = E$ (i.e., the subscripts are to be chosen independently). On the other hand, $p(f_d,g_d)$ will always mean $\{p(f_d,g_d); d \in D\}$; the domain of the net has only one factor.

1.2. FILTERS. A *filter base* in a set X is a nonempty collection of nonempty subsets of X such that the intersection of two members contains a member of the collection. A *filter* is a filter base which in addition contains every superset of a set it contains. We shall occasionally say "filter" instead of "filter base" when the distinction is insignificant. If A is a subset of X, $\{B: A \subset B \subset X\}$ is the filter *based on* A; $\{B: x \in B \subset X\}$ is *based on* x. However, a filter not based on any set need have no smallest set. If for two filters $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F} \subset \mathcal{G}$, then $\mathcal{G}$ is *finer* than $\mathcal{F}$ ($\mathcal{G}$ contains more sets; it may be thought of as a "subfilter" of $\mathcal{F}$). Clearly the intersection of two (or finitely many) filters is again a filter, having fewer sets (coarser) than either initial filter. A filter is said to be *eventually in* A if A is a set of the filter; by a stretch of terminology, a filter $\mathcal{F}$ is *frequently in* A if A intersects each set of $\mathcal{F}$.

A class of filters may in fact contain the intersection of any two filters in it and every filter finer than any filter in it; to avoid confusion with a simple filter, such a class will be called an *ideal*.

Let $\mathcal{F}_i$ be a filter in X_i for i in some index set. The *product filter* Π_i in ΠX_i has as base the collection of sets of the form ΠF_i, where each F_i comes from $\mathcal{F}_i$ and $F_i = X_i$ except for finitely many i.

1.3. RELATIONSHIPS BETWEEN NETS AND FILTERS. (a) Let $(D,\geq)$ be a directed set and for each d in D let $D_d = \{g \in D: g \geq d\}$. Then if $f \geq d$, $f \geq e$, we have $D_f \subset D_d \cap D_e$ so the sets D_d form a filter base in D. Conversely, if $\mathcal{F}$ is a filter base, $\{\mathcal{F},\subset\}$ is a directed set, since two elements of $\mathcal{F}$ are followed by their intersection. (b) Let $\{S,D\}$ be a net in X and consider the sets $S(D_d) = \{S_g: g \geq d\}$. These sets form a filter base in X; the filter of supersets is called the *filter associated with* S. If T is a subnet of S, range T $\subset$ range S and it will be shown that the filter associated with T is finer than that associated with S. Conversely, let $\mathcal{F}$ be a filter in X and for each $F \in \mathcal{F}$ choose some point $f_F \in F$; then $\{f_F,\mathcal{F},\subset\}$ is a net in X. This net (which is not unique) is called a *net associated with* $\mathcal{F}$. It is eventually in each set of $\mathcal{F}$, since $G \subset F$ implies $f_G \in G \subset F$. Similarly, any net is eventually in each set of the filter associated with it. (c) For any filter $\mathcal{F}$ we may construct a net S such that $\mathcal{F}$ is the filter associated with S; this illustrates a complicated net construction. Let D be the set of all doubles (x,F) such that F is in $\mathcal{F}$ and x is in F. Define $(x,F) \geq (y,G)$ iff $F \subset G$. D is directed: $(z,F \cap G)$ follows (x,F) and (y,G).

Now define the function $S(x,F) = x$; S is a net on D into X. Further, for any (x,F) in D, $D_{(x,F)}$ includes at least all points of the form (y,F) for $y \in F$ and thus $S(D_{(x,F)}) = F$. Thus, the filter associated with S is $\mathcal{F}$. (d) An *ultrafilter* is a filter which contains either A or the complement of A for each subset A of X. A *universal net* is a net which is eventually in A or its complement for each subset A of X. The net associated with an ultrafilter is universal, and the filter associated with a universal net is an ultrafilter. (e) Let X be a set with only one point. The only possible filter on X contains X as its single set. There are however infinitely many nets in X, one for each directed set. Thus knowing the range of a net tells little about its domain; in this sense nets are more general than filters.

1.4. LEMMA. *If $\mathcal{F}$ is not an ultrafilter, there is a filter strictly finer than $\mathcal{F}$.*

Suppose neither A nor CA (the complement of A) is in $\mathcal{F}$. Each set in $\mathcal{F}$ must intersect A, since otherwise one would be a subset of CA and $\mathcal{F}$ would contain CA. Let $\mathcal{F}_A = \{A \cap F\colon F \in \mathcal{F}\}$. This is clearly a base for a filter which contains each set of $\mathcal{F}$ as well as the set A, and thus is strictly finer than $\mathcal{F}$.

1.5. THEOREM. *Every filter $\mathcal{F}$ is contained in an ultrafilter.*

Consider the class of filters finer than $\mathcal{F}$; it is nonempty (containing $\mathcal{F}$). Let $\mathcal{F}_a, \mathcal{F}_b, \mathcal{F}_c, \cdots$ be a nest (finite or infinite) of such filters (i.e., be linearly ordered by $\subset$); for finitely many sets $F_1, \cdots, F_n$, each in some $\mathcal{F}_i$, all F_k are in the largest $\mathcal{F}_i$ and hence their intersection is (so the intersection is nonempty). Hence, $\{F_i; F_i \in \mathcal{F}_a$ for each i and some $a\}$ is a base for a filter which is an upper bound for the nest. We may now apply Zorn's Lemma to assert that there is at least one maximal filter in the class of filters finer that $\mathcal{F}$; by 1.4, this filter must be an ultrafilter.

1.6. LEMMA. *Let $\mathcal{F}$ be a filter and S a net in X, and suppose S is frequently in each set of $\mathcal{F}$. Then S has a subnet which is eventually in each set of $\mathcal{F}$.*

$(\mathcal{F},\subset)$ is directed; denote S by $\{S_d, D, \geq\}$. Let E be the set $\{(m,F)\colon m \in D, F \in \mathcal{F}, S_m \in F\}$ as directed by the product order $\geq\subset$ of $D\times\mathcal{F}$; to show this is a directed set, suppose (m,F) and (n,G) are in E, let $p \geq m,n$ and $q \geq p$ s.t. $S_q \in F \cap G$. Then $(q, F \cap G)$ is the desired upper bound. Define on E $P(m,F) = m$; P is eventually beyond each d in D, so $S\cdot P$ is a subnet of S. $S\cdot P$ is eventually in each F in $\mathcal{F}$, since if S_m is in F, $(n,G) \geq\subset (m,F)$ implies $S\cdot P(n,G) = S_n \in G \subset F$.

1.7. THEOREM. *Every net has a universal subnet.*

Consider the class of filters $\mathcal{F}$ such that the net S is frequently in each set of $\mathcal{F}$. By the argument of 1.5, this class has a maximal element. Now we apply 1.4 to show that this maximal filter is an ultrafilter. Call the maximal filter $\mathcal{F}$ and suppose $\mathcal{F}$ contains neither A nor CA for some $A \subset X$. S is then frequently in each set of $\mathcal{F}_A = \{F \cap A\}$ or each set of $\mathcal{F}_{CA}$; if not, S is eventually outside of some $F \cap A$ and eventually outside some $G \cap CA$, hence eventually outside $(F \cap A) \cup (G \cap CA) \supset F \cap G$, a contradiction since $F \cap G \in \mathcal{F}$ (and by hypothesis S is frequently in each set of $\mathcal{F}$). Thus the supposition that $\mathcal{F}$ is maximal and that $\mathcal{F}$ contains neither A nor CA conflict, so $\mathcal{F}$ must be an ultrafilter. Then by Lemma 1.6 S has a subnet which is eventually in each set of $\mathcal{F}$, and by 1.3(d) that subnet is universal.

1.8. THEOREM. *Let* $T = S \cdot N$ *be a subnet of the net* S *in* X *and denote by* $\mathcal{F}(S)$ *and* $\mathcal{F}(T)$ *the associated filters. Then* T *is eventually in each set of* $\mathcal{F}(S)$*; that is,* $\mathcal{F}(T)$ *is finer than* $\mathcal{F}(S)$.

Every set of $\mathcal{F}(S)$ contains a base set, of form $\{S_g: g \geq d \in D_S\}$. There is some n in $D_N = D_T$ such that for $m \geq n$, $N_m \geq d$, and hence $T_m = S \cdot N_m \in \{S_g: g \geq d\}$. Thus T is eventually in that set, and that set must contain a basis set of $\mathcal{F}(T)$ and thus belong to $\mathcal{F}(T)$.

1.9. THEOREM. *Let* T *and* S *be nets in* X *and* $\mathcal{F}(T)$ *be finer than* $\mathcal{F}(S)$*. Then* T *has a subnet which is also a subnet of* S. (In general T is not a subnet of S since its range may not be a subnet of that of S).

Let S have domain $(D,\geq)$, T domain $(E,>)$. Let $F = \{(d,e) \in D \times E: T_e = S_d\}$. We must show that F is directed by the product order $\gg$ of $D \times E$. In fact, suppose (d,e) and (f,g) are in F; let $h \geq d,f$ and $k > e,g$. By hypothesis there is $m > k$ such that $T_m \in \{S_n: n \geq h\}$, i.e., for some $n \geq h$, $T_m = S_n$. Thus (n,m) is in F and follows (d,e) and (f,g), so F is directed. Now let $P_1(d,e) = d$, $P_2(d,e) = e$, so that $S \cdot P_1(d,e) = S_d = T_e = T \cdot P_2(d,e)$ throughout F. It is clear that P_1 is eventually beyond each point of D; similarly for P_2. Hence $S \cdot P_1$ and $T \cdot P_2$ are subnets of S and T which are identical (i.e., have the same domain and same values on it).

1.10. THEOREM. *Let* S *be a net,* $\mathcal{F}$ *and* $\mathcal{G}$ *filters in* X*. Suppose* S *is frequently in each set of* $\mathcal{F} \cap \mathcal{G}$*. Then some subnet of* S *is eventually in each set of* $\mathcal{F}$ *or eventually in each set of* $\mathcal{G}$.

For each $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $F \cup G$ is in $\mathcal{F} \cap \mathcal{G}$ so S is frequently in $F \cup G$. Thus S is frequently in F or frequently in G; suppose it fails to be frequently in F. then for every $G_1 \in \mathcal{G}$, S is frequently in $F \cup G_1$, hence in G_1. Thus S is frequently in every $F \in \mathcal{F}$ or frequently in every $G \in \mathcal{G}$, so we may apply Lemma 1.6 to find a subnet eventually in each set of one or the other filter.

1.11. THEOREM. *A filter is the intersection of all ultrafilters finer than it.*

Clearly if a set is in the filter $\mathcal{F}$, it is in all finer ultrafilters. Conversely, let A be in each ultrafilter but not in $\mathcal{F}$; then each F in $\mathcal{F}$ intersects CA so $\{F \cap CA: F \in \mathcal{F}\}$ is a base for a filter finer than $\mathcal{F}$. Picking any ultrafilter finer than it, we get an ultrafilter finer than $\mathcal{F}$ which does not contain A, a contradiction. Thus $\mathcal{F}$ must contain A.

2. LIMIT SPACES

2.1. DEFINITIONS AND REMARKS. Historically limit spaces have been defined in terms of sequences (e.g., Frechet, Kuratowski) and more recently in terms of filters (e.g., Choquet, Fischer). The following will be based largely on nets, but some attempt will be made to connect net-based with filter-based theory.

A *convergence* for a set X is a collection C of ordered pairs (S,x) where S is a net in X and x is a point in X. Then x is a *limit* of S(C), S converges to x(C) or $S \to x(C)$. The pair (X,C) will be called a *limit space;* often reference to C will be omitted. Note that this definition, allowing an arbitrary set of ordered pairs, allows extremely wild notions of convergence; "no net converges" or "a constant net converges to every point" are examples.

Usually the minimum restrictions desired are those placed on an *$\mathcal{L}$ space* by Frechet (customarily for sequences): (a) any net whose range is {x} (a constant net) converges to x, and (b) a subnet of a convergent net converges to any limits the net does. (Either may of course also have other limits). Kuratowski (also working with sequences) called an $\mathcal{L}$-space an *$\mathcal{L}^*$-space* if (c) if a net fails to converg to x, then it has a subnet no subnet of which converges to x. Fischer developed extensively two filter-based spaces, the more restricted of which was a simple neighborhood structure: a space is $\mathcal{T}_1$ if (1) for each point x in X, there is a filter $N(x)$ of supersets of {x}, called a *system of neighborhoods* of X, such that the net converges to x if and only if it is eventually in each neighborhood of the system. (Fischer had as convergent filters those finer than the neighborhood system). It should be noted that the notation "$\mathcal{T}_1$" is nonstandard.

Fischer's broader axiom required merely that the convergent filters form an ideal: the space is a *$\mathcal{T}$ space* if it satisfies (2) for each point x in X, there is an ideal $N(x)$ of filters $N(x)$ (called *pseudo-neighborhood system*) such that a net converges to x if and only if there is some pseudo-neighborhood system $N(x)$ such that the net is eventually in each pseudo-neighborhood of the system. In particular, $N(x)$ must contain the filter of supersets of {x} (this guarantees that the constant nets converge). Choquet, working with filters, proposes to initially determine convergence only for ultrafilters and then to let an arbitrary filter converge to x if each finer ultrfilter does. An equivalent definition for nets (since a universal net can have proper subnets) would be to let a universal net converge to x whenever any subnet did, and then to let an arbitrary net converge whenever every universal subnet of it does. After the rest of this section, the reader should have little difficulty in relating Choquet convergence to $\mathscr{L}^*$ convergence.

2.2. ELEMENTARY RELATIONS. The rest of this section will be devoted to establishing implications between the above axioms and introducing and relating to them the notion of a topological space. We begin with some elementary examples. (a) Let X be nonempty and C void. X is a limit space but is not $\mathscr{L}$. (b) Let X be the real line R; let constant nets converge and the sequence 1, 2, 3, ··· together with its subnets (which are in general not sequences) converge to zero. This space is $\mathscr{L}$; it is not $\mathscr{L}^*$ or $\mathscr{J}$ since the sequence 0, 1, 2, ··· fails to converge (each subnet of it has a subnet in common with 1, 2, 3, ··· which thus converges to 0, so X is not $\mathscr{L}^*$, and 0, 1, 2, ··· is eventually in any hypothetical pseudo-neighborhood that 1, 2, 3, ··· is eventually in, so X is not $\mathscr{J}$). (c) It is clear that a $\mathscr{J}_1$ space is a $\mathscr{J}$ space; the ideal consists of the neighborhood filter together with all finer filters. A $\mathscr{J}$ space is an $\mathscr{L}$ space since (a) constant nets converge and (b) a subnet eventually is in any pseudo-neighborhood the net is eventually in. (d) A $\mathscr{J}_1$ space is an $\mathscr{L}^*$ space; the only difficulty arises with axiom 2.1 (c). If a net fails to converg to x, it is frequently outside some neighborhood of x and thus has a subnet which is eventually outside that neighborhood; any sub-subnet would also remain outside the neighborhood and thus fail to converge. An $\mathscr{L}^*$ space is an $\mathscr{L}$ space by definition.

2.3. EXAMPLE. *A $\mathscr{J}$ space need not be $\mathscr{L}^*$.*

Define N to be a class containing all ultrafilters and all finite intersections of ultrafilters. Let X be the real line and let a net converge to x whenever its range is {x} and to 0 whenever it is eventually in each set of some filter in N. N clearly contains finite intersections of its members; it contains filters finer than its members since a filter finer than a finite intersection of ultrafilters is merely an intersection of some of those ultrafilters (this may be proved rigorously, using 1.10 and 1.11). It is also clear that the trivial filter {X} is not a finite intersection of ultrafilters (to show it is not an intersection of n ultrafilters, divide the line into n+1 disjoint parts; each filter contains exactly one of those parts).

Now let D be the set of ordered pairs of real numbers (r,s), with order $(r,s) \geq (p,q)$ iff $s \geq q$, and define $P(r,s) = r$. The net P is frequently in every set, so it does not converge to 0; however, every subnet of P has a universal subnet, which must converge to 0. The space is therefore $\mathcal{J}$ but not $\mathcal{L}^*$.

2.4. EXAMPLE. *An $\mathcal{L}^*$ space need not be $\mathcal{J}_1$.*

Again let X be the real line, constant nets converge, and also let any net which is eventually in some finite set converge to 0 (in particular, any constant net converges to 0). If X were $\mathcal{J}_1$, each neighborhood of 0 would have to contain each point of the line (since constant nets converge to 0), an absurdity. On the other hand, the convergence is clearly $\mathcal{L}^*$ with respect to points other than 0, and if a net S fails to converge to 0 we need only ncte that the collection of complements of finite sets is a filter. Then S is frequently in each set of the filter, so it has a subnet which is eventually in each set of the filter; clearly no sub-subnet can converge to zero. Hence, X is $\mathcal{L}^*$ but not $\mathcal{J}_1$.

2.5. THEOREM. *An $\mathcal{L}^*$ space is always a $\mathcal{J}$ space* (this gives at least a partial notion of the implications of the "subnet-of-a-subnet" requirement).

It is necessary for us to exhibit the class $N(x)$ of preudo-neighborhood systems at x. For each net S converging to x, denote the associated filter by $F(S)$. Now consider the class of all such $F(S)$, and show it is the desired $N(x)$. Clearly if a net T converges to x, it is eventually in each set of a psuedo-neighborhood system $F(T)$; conversely, if a net T is eventually in each set of $F(S)$ (or any finer filter), then by THEOREM 1.8 any subnet of T has an associated filter finer than that of S and thus by THEOREM 1.9 has a subnet in common with S. That subnet converges to x since S does, and thus since the space is $\mathcal{L}^*$ T must converge to x. Finally, the intersection of two members of $N(x)$ is a member by application of THEOREM 1.10 and the fact that a net can be found in any filter whose associated filter is the original one (by 1.3 (c)).

2.6. SUMMARY. We have now given proofs and counterexamples to establish a conclusive order among a number of axiom systems on a limit space. In order of increasing restrictiveness are limit spaces, $\mathcal{L}$ spaces, $\mathcal{J}$ spaces, $\mathcal{L}^*$ spaces, and $\mathcal{J}_1$ spaces; each condition implies those preceding it and none implies any following it. In the following some attempts have been made to use weaker rather than stronger spaces in hypotheses, but it is not asserted that the weakest available space has been used (this writer is in some cases uncertain).

2.7. REMARKS AND DEFINITIONS. We now examine some restrictions on limit spaces which are usually not directly expressible in terms of nets or filters. In particular, we develop what are ordinarily taken as the basic concepts of point set topology, e.g., open and closed sets and closure operations.

A point x will be said to be *interior* to a set A if any net converging to x is eventually in A. Thus x is interior to any neighborhood of x in a $\mathcal{T}_1$ space (but not necessarily to a pseudo-neighborhood); in any space, A will loosely be called a *neighborhood* of x. A set all of whose points are interior is called *open*; a set which contains every limit of all nets in the set is called *closed*. A set X may have nets in it converging to points outside X; the set taken together with such points (if any) is the *closure* of X, denoted $\overline{X}$. A set is closed if and only if it is equal to its closure; it should be noted that in general the closure of X is not itself a closed set.

2.8. EXAMPLE. *In a $\mathcal{T}_1$ space, the closure of a set need not be closed.*

Let X be the set of real numbers. For x real and r positive and rational, let $K_r(x)$ be the set of all real y such that $|x-y| < r$ and x-y is irrational or zero. Call the collection of all $K_r(x)$ for fixed x a base for the neighborhood filter $N(x)$; the result is a $\mathcal{T}_1$ space, in which no not-eventually-constant net of rational numbers converges to a rational. Let A be the set of rationals in (0,1); then $\overline{A} = (0,1)$ but $\overline{\overline{A}} = [0,1]$, so $\overline{A}$ is not closed.

It is interesting that the operation of "weak linear closure", due to a theorem of Nikodym, leads to a $\mathcal{T}$ space in which even countably many closures need not yield a closed set.

2.9. THEOREM. *Any union of open sets is open; the intersection of two open sets is open* (in any limit space).

Let A_i be any union of open sets, and let x be a point of the union; then x is in one of the A_i, so a net converging to x is eventually in A_i and thus in the union. For the second statement, let A and B be open and x an element of their intersection. A net converging to x is eventually in A and eventually in B, hence eventually in their intersection.

2.10. THEOREM. *In any limit space, the complement of an open set is closed. In an $\mathcal{L}$ space, the complement of a closed set is open.*

Let A be open; denote its complement CA. A net entirely in CA cannot converge to a point of A since it fails to be eventually in A; thus any limit of the net is in CA and CA is closed. Conversely, let A be closed and let S be a net converging to a point x of CA. If S is frequently in A, it has a subnet eventually in A which also converges to x, impossible since A is closed. Thus S must eventually be in CA.

2.11. THEOREM. *In a* $\mathcal{T}_1$ *space,* x *is in* $\overline{A}$ *if and only if each neighborhood of* x *intersects* A.

A net S in A converges to x; S is eventually in each neighborhood so each neighborhood meets A. $(N(x),\subset)$ is a directed set; for each N in $N(x)$ let $x_N \in N \cap A$. Then the net $\{x_N, N(x), \subset\}$ converges to x so $x \in \overline{A}$.

2.12. THEOREM. *In a* $\mathcal{T}_1$ *space the following are equivalent: (a) for any subset* A *of* X, $\overline{A} = \overline{\overline{A}}$, *and (b) the open neighborhoods of* x *form a base for the filter* $N(x)$.

(a) implies *(b)*: By 2.9 the intersection of open neighborhoods is open. We must show that an arbitrary neighborhood N of x contains an open neighborhood. $x \in N$ so $x \notin CN$ and $x \notin \overline{CN}$. But $\overline{CN}$ is closed so $C\overline{CN}$ is open. It contains x and thus is an open neighborhood; it is a subset of CCN = N. *(b)* implies *(a)*: Let $x \in \overline{\overline{A}}$. Any neighborhood N of x contains an open neighborhood M; by 2.11, M intersects $\overline{A}$ and we may pick a point y in that intersection. Since M is open it contains a set of $N(y)$, hence it intersects A. Thus x is in $\overline{A}$, so $\overline{A} = \overline{\overline{A}}$ and is closed.

2.13. DEFINITIONS AND REMARKS. The conditions of THEOREM 2.12 are exactly the requirements for a topological space. More often, a topological space is defined by giving a class t of subsets of X such that an arbitrary union or finite intersection of sets of t is again in t. One then defines a net as converging to x whenever the net is eventually in each set of t which contains x. Once it is shown that the sets of t are exactly the open sets of the resulting $\mathcal{T}_1$ space, 2.12 (b) will hold and the equivalence will be complete.

2.14. THEOREM. *Let* X *be a set and* t *a class of subsets. Under the conditions of* 2.13, t *is the class of open sets in the* $\mathcal{T}_1$ *space* X.

Each set of t is open since it is a neighborhood of each of its points. If A is any open set and x is in A, there is a set of t which contains x and is contained in A; the union of these sets for every x in A is equal to A and belongs to t, so A is in t.

We will occasionally use the notation t_C for a topology arising from a convergence as in 2.12 - 2.13, and C_t for a convergence arising from a topology. (X,t), where t is the class of open sets, will denote a topological space.

2.15. REMARKS. THEOREM 2.12 provides a reasonably useful set of necessary and sufficient conditions that a space be topological. Kelly provides another set of conditions which will apply to a space which is merely $\mathcal{L}^*$; the bulkiness of the condition (the iterated limit property) as well as the difficulty in deciding that a space is $\mathcal{L}^*$ make it sufficient here to merely state the result, which will not be used in the sequel.

2.16. THEOREM. (Theorems 2.4 and 2.9, Kelley, *General Topology*). *An* $\mathcal{L}^*$ *space is topological if and only if it satisfies the following iterated limit condition: Let* D *be a directed set and* E_m *a directed set for each* m *in* D. *For* $(m,f) \in D \times \prod_m E_m$ *define* $R(m,f) = (m,f_m)$ *where* f_m *is the* m^{th} *coordinate of* f. *Let* $S(m,n)$ *be defined for* n *in* E_m. Then whenever $\{S(m,n), n \in E_m\}$ *converges to* x_m *for all* m *in* D *and also* $\{x_m, m \in D\}$ *converges to* x, *it follows that* $S \cdot R(m,f)$ *converges to* x.

2.17. DISCUSSION AND EXAMPLE. Diagram 1 may be of some help. If one

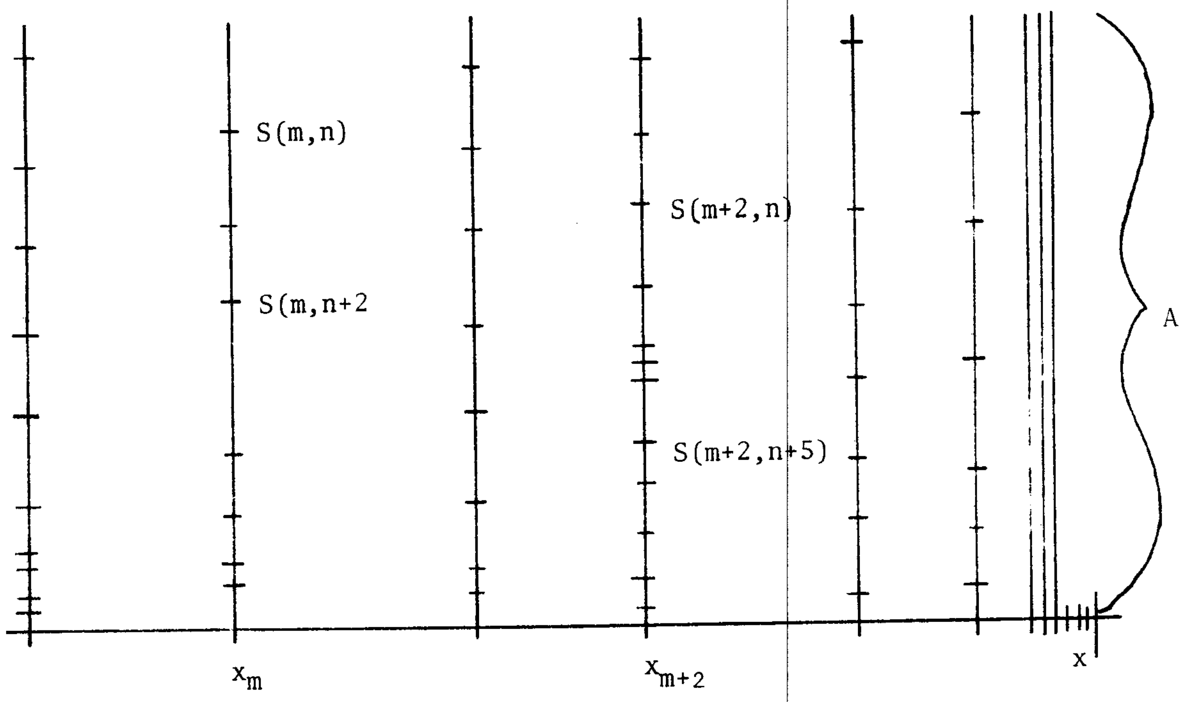

DIAGRAM 1

thinks of the points $S(m,n)$ as being in a set A, the points x_m as limits of nets in A are in $\overline{A}$, and the point x is in $\overline{\overline{A}}$. The iterated limit property produces a specific net in A converging to x, so it implies that $\overline{A} = \overline{\overline{A}}$ but is somewhat stronger, requiring in addition that a specific net converge. EXAMPLE 2.4 (nets eventually in a finite set converge to 0) is an $\mathcal{L}^*$ space and the closure of any set is closed (the closure of the empty set is empty, and of any nonempty set X is $X \cup \{0\}$). It is not $\mathcal{T}_1$ and thus not topological, so the iterated limit property cannot hold.

2.18. EXAMPLE. Convergence almost everywhere. We shall attempt to use as "standard examples" spaces of functions from the reals into the reals. Suppose we consider the class of all real-valued functions on [0,1] and specify that a net of such functions converges if and only if it converges pointwise except on a set of measure zero (a few elements of measure theory as it is used here are outlined in an appendix). This is clearly a $\mathcal{T}$ space if the definition is stated as follows: f_d converges to f(a.e.)if and only if for some set of measure zero A, f_d is eventually in each set of the form

$$N(x,\varepsilon) = \{f_d\colon |f_d(x) - f(x)| < \varepsilon\}\ x \notin A,\ \varepsilon > 0.$$

On the other hand, the following argument will show that (X,a.e.) is not $\mathcal{T}_1$ and thus not topological. Suppose a neighborhood basis could be found for some point x. There exists a sequence converging to x in measure (see 4.16 or the appendix, to be published in the next issue of this journal) which fails to converge a.e.; i.e., there is a neighborhood U which the sequence is frequently outside. We may thus find a subsequence which remains outside U; it will still converge in measure. Now, by a theorem of which a proof is outlined in the appendix, we may find a subsequence of this last sequence which does converge to x a.e., contradicting the fact that it remains outside the neighborhood. This contradiction shows that it is impossible to construct a neighborhood structure for (X, a.e.). .

We have thus found a space of some mathematical application which is not a topological space; this will serve to justify some study of general limit spaces. We omit here the somewhat less direct proof that (X,a.e.) is not $\mathcal{L}^*$.

3. GENERAL TOPOLOGY IN LIMIT SPACES.

As noted in the introduction, it has been decided to present a few elementary theorems from a number of topics of general topology, rather than to attempt to develop ane topic to great depth or with weakest possible hypotheses. Most theorems are stated for fairly weak spaces, since the corresponding theorems for topological spaces are all readily available. To a large extent the following discussion will follow the usage of Bourbaki.

3.1. DEFINITIONS. If the convergence C_1 on X is smaller than the convergence C_2 (i.e., fewer nets converge) we call C_1 *finer* than C_2 and C_2 *coarser* than C_1.

3.2. THEOREM. *If* C_1 *is finer than* C_2, *the space* (X,C_2) *has fewer open sets than* (X,C_1). *If* (X,t) *and* (X,s) *are topological and* t *is a subset of* s, *the convergence* C_s *is finer than the convergence* C_t.

Suppose a set is open with respect to C_2, and a net S converges to a point x of the set (C_1). Since C_1 is a subset of C_2, S converges C_2 and is eventually in the set which must therefore must be open in (X,C_1). Now let t be a subset of s (this theorem allows us to call t *coarser*, s *finer*) and suppose S converges to x in (X,s). Then S is eventually in each set of s containing x, so it must eventually be in each set of t containing x and converge in (X,t).

3.3. REMARK. It is worth noting that if an arbitrary $\mathcal{L}$ space (X,C) is given, a class of open sets (denoted by t_C) is determined and this may be used to define a convergence, denoted by C*. A net convergent C is eventually in each required open set and so converges C*, but if C is not topological itself C* will be strictly coarser. In any case, C* will be the finest topological convergence coarser than C, since if C' is coarser than C it will have fewer open sets than (X,C) and thus no fewer convergent nets than (X,C*).

3.4. DEFINITIONS. Let $f:(X,C) \to (X',C')$ be a mapping between two limit spaces. f is *continuous at* x if whenever the net S converges to x, the net f·S converges to f(x). f is *continuous on* a set if it is continuous at each point of the set. X and X' are *homeomorphic* if there is a 1-1 continuous onto mapping whose inverse is continuous; the mapping is a *homeomorphism*.

3.5. THEOREM. *Let* X *and* X' *be arbitrary limit spaces; then* (a) *implies* (b) *implies* (c). *If* X' *is topological (*X *need not be) then* (c) *implies* (a).

(a) f *is continuous*.

(b) *for any subset* A *of* X, $f(\bar{A}) \subset \overline{f(A)}$.

(c) $B \subset X'$ *is open (closed)*, $f^{-1}(B)$ *is open (closed)*.

(a) implies (b): Let $x \in \bar{A}$; i.e., a net S in A converges to x. Then f(S) is a net in f(A) converging to f(x), so $f(x) \in \overline{f(A)}$. (b) implies (c): Let B be closed. Let $x \in \overline{f^{-1}(B)}$, so $f(x) \in f(\overline{f^{-1}(B)}) \subset \overline{ff^{-1}(B)} \subset \bar{B} \subset B$, hence $x \in f^{-1}(B)$, so $\overline{f^{-1}(B)} \subset f^{-1}(B)$ and $f^{-1}(B)$ is therefore closed. (c) implies (a): Suppose X' is topological and S converges to x in X. For any open neighborhood N of f(x), $f^{-1}(N)$ is an open set containing x. Thus S is eventually in $f^{-1}(N)$, so f(S) is eventually in N and converges to f(x). Since each image of a convergent net converges, f is continuous. (It is clear that the open and closed set conditions are equivalent, by set complementation).

3.6. DEFINITIONS. Let X_i be a limit space for each i in I, and $\prod_I X_i$ the Cartesian product. The *product convergence* is the coarsest convergence on the Cartesian product X such that each of the projection mappings $P_i: X \to X_i$ is continuous. That is, a net in X converges if each of its projections converges to the image of the proposed limit. It should be noted that a net in the product space is not in general a product of nets in the X_i.

It will be useful occasionally to use rigorously the notion of subspace. If A is a subset of the limit space X, A may be regarded as a limit space by considering as convergent those nets in A which converge to points of A in the limit structure on X. It is clear that the various elementary restrictions on X ($\mathcal{L}$, $\mathcal{T}$, $\mathcal{L}^*$, $\mathcal{T}_1$) will carry over to A; it should be noted however that a set open or closed in A need not be open or closed in X.

3.7. LEMMA. *Let* X *be any factor of a product limit space* X', *such that for some* x *in* X' *each constant net in* {x} *converges to* x. *Then* X' *has a subspace homeomorphic to* X.

Let X(x) denote the set of points of X' which have one coordinate (the one coming from X) arbitrary and all other coordinates as in x. The restriction of the X^{th} projection map, $P/X: X(x) \to X$ then is continuous and has a unique inverse; we must show that the inverse is continuous. Let S converge to s on X and denote their images under P^{-1} by $P^{-1} \cdot S$ and $P^{-1}(s)$. It is clear that the X^{th} projection of the net $P^{-1} \cdot S$ is S, and any other projection is constant and thus convergent (since each constant net in x converges, any projection of such a net does); hence $P^{-1} \cdot S$ converges and the homeomorphism is established.

3.8. THEOREM. *If a product space is nonempty and* $\mathcal{L}$ ($\mathcal{T}$, $\mathcal{L}^*$, $\mathcal{T}_1$, topological) *then each factor is.*

By the remark of 3.6, any subspace of the product retains the desired property; by 3.7 the factor is homeomorphic to a subspace and the property carries to it.

While in many cases restrictions on factors carry over to restrict the product space, a fairly general proof like that above seems difficult. We shall prove only two such theorems here.

3.9. THEOREM. A *product of* $\mathcal{T}_1$ *spaces is* $\mathcal{T}_1$.

We shall show that the collection of all Π_i such that N_i is a neighborhood of x_i in X_i, with all but finitely many $N_k = X_k$, is a neighborhood basis for $x = \Pi x_i$. It is clear that if S converges to x in this neighborhood system, $P_k \cdot S$ is eventually in $P_k(\Pi N_i) = N_k$, i.e., $P_k \cdot S$ converges to x_k in the limit space X_k. Similarly, if $P_k \cdot S$ converges to x_k in X_k for each k, $P_k \cdot S$ is eventually in each N_k, $P_j \cdot S$ eventually in N_j, $\cdots$, and for K a finite set $\prod_K P_k \cdot S$ is eventually in $\prod_K N_k$. Letting all other $N_j = X_j$, $\Pi P_k \cdot S$ and hence S are eventually in each ΠN_k and converge to x in terms of the neighborhood system.

The above neighborhood system may be compared with the definition of product filter given in 1.2; the neighborhood filter in the product space is the product of the neighborhood filters in the factors.

3.10. THEOREM. A *product of topological spaces is topological.*

By 3.9 the space is $\mathcal{T}_1$ and we have a neighborhood basis at each point. Let V be an arbitrary neighborhood of x; by 2.12, it will suffice to show that V contains an open neighborhood of x. V contains a basis neighborhood ΠV_i where each V_i is a neighborhood of x_i in X_i (and all but finitely many $V_i = X_2$). For each V_i let U_i be an open subneighborhood, equal to X_i when V_i is. It is then clear that ΠU_i is a neighborhood of each of its points, hence open, is a neighborhood of x, and is contained in ΠV_i and thus in V. This completes the proof.

We remark again that the carrying over to products of conditions such as $\mathcal{L}^*$, $\mathcal{J}$, and the requirement that closures of sets be closed is non-trivial and is omitted primarily due to space limitations.

3.11. DEFINITIONS. Separation axioms lend themselves especially well to treatment in terms of convergence. We may define a space to be T_1 (not to be confused with $\mathcal{J}_1$) if no net which is eventually in {x} converges to any point $y \neq x$. A space is T_2 or Hausdorff if and only if no net has two limit points. Note that a T_2 $\mathcal{L}$ space is automatically T_1. It is also clear (by examining various projections of a hypothetical doubly convergent net) that a product of T_2 limit spaces is T_2; the converse requires the factors to be $\mathcal{L}$ spaces.

3.12. THEOREM. *A $\mathcal{J}_1$ space is T_2 if and only if each two points have disjoint neighborhoods.*

If x and y have disjoint neighborhoods, a net S cannot eventually be in both of those neighborhoods. If on the other hand every neighborhood V of x and every neighborhood U of y intersect, define for each (V,U) a point g(V,U) in $V \cap U$. Then the net $\{g(V,U), N(x) \times N(y), \subseteq \subseteq\}$ converges to both x and y and the space is not T_2.

3.13. THEOREM. *Let* f,g *be two continuous mappings from any limit space* (X,C) *into a* T_2 *limit space* (Y,D). *Then* $A = \{x : f(x) = g(x)\}$ *is closed in* X.

Suppose some net S in A converges to a point y(C). By continuity $f \cdot S$ converges to f(y)(D), $g \cdot S$ converges to g(y)(D). But as nets (since S is in A) $f \cdot S$ and $g \cdot S$ are identical and have at most one limit point $f(y) = g(y)$, and y is in A so A is closed.

3.14. DEFINITION. (X,C) is *compact* if and only if each universal net converges. Since each net has a universal subnet, we might wish merely to require that each net have a convergent subnet. This condition does not however in general imply compactness, since a universal net may have a proper subnet (which is then also universal and in fact has the same associated filter).

3.15. THEOREM. *Let* X *be a* $\mathcal{J}$ *space. Then* X *is compact if and only if each net has a convergent subnet.*

Sufficiency of compactness was noted above for any limit space. Conversely, choose a universal net; it has a convergent subnet. The universal net is eventually in each set the subnet is in, so it is eventually in each necessary pseudo-neighborhood and converges.

3.16. THEOREM. *Let X be any compact limit space. Then any open covering of X has a finite subcovering.*

Let $\mathcal{A}$ be an open covering, and suppose that for any finite subset of $\mathcal{A}$ (consisting of sets A, B, C, ···, K) we can find a point x(A, B, C, ···, K) which is outside all of them. Then x is a net whose domain is the set of finite subsets of $\mathcal{A}$, ordered by $\subset$; it has a subnet converging to a point z. But then no set of $\mathcal{A}$ can contain z since if one did the net would stay eventually outside that open set about z and fail to converge to z. This contradicts the supposition that no finite subcovering exists.

3.17. THEOREM. *Let X be a topological space in which each open covering has a finite subcovering. Then X is compact.*

Let S be a net with no convergent subnet, and for each d in the domain D of S let $D_d = \{e: e \geq d\}$. Let $A_d = \overline{CS(D_d)}$, an open set. The intersection of finitely many D_d contains another, no finite collection of sets A_d covers X; hence no collection does. Let y be outside of every A_d, i.e., in each $\overline{S(D_d)}$; hence each neighborhood of y intersects each $S(D_d)$, and a subnet of S may be found convergent to y. Then by 3.15 X is compact.

3.18. THEOREM. (Tychonoff) *Any product of compact limit spaces is compact.*

Consider a universal net in the product space; any projection is universal and thus converges, so the original net converges.

This is so general due to the choice of definition in 3.14. Had the convergent subnet definition been chosen, the proof of 3.15 would be essential to the above manner of proof, which would then apply only to $\mathcal{T}$ spaces.

3.19. THEOREM. *A space finer than a Hausdorff space is Hausdorff. A space coarser than a compact space is compact.*

Immediate from the definitions; a coarser space has more convergent nets.

3.20. DEFINITIONS AND REMARKS. One type of question apparently quite unsuited to study in terms of nets is countability. In the ordinary definition, a topological space is *first countable* if there is a countable basis for the neighborhood filter at each point; we retain this definition for $\mathcal{T}_1$ spaces, and it might even be salvageable in $\mathcal{T}$ spaces. On the other hand, it is sometimes useful to consider conditions relating to sequences. We shall call a space X *sequential* if whenever a point x is in the closure of a set A, there is a sequence in A converging to x. (One may study spaces in which the only items of concern are sequences; but if the space is not by nature first countable much machinery will not apply since sequences have subnets which are not sequences).

We omit entirely here discussion of second countable spaces (spaces with a countable collection of open sets generating the topology).

3.21. THEOREM. *Let X be a first countable $\mathcal{T}$ space. Then X is sequential.*

Suppose a net in A converges to x; thus A intersects each set in some pseudo-neighborhood system for x, and since that system has a countable base the construction of a net in A converging to x (2.11) will yield a sequence.

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