Journal of

UNDERGRADUATE MATHEMATICS

CONTENTS

VOLUME 2
NUMBER 1

MARCH
1970

CONVERGENCE AND ABSTRACT SPACES IN FUNCTIONAL ANALYSIS[1]

Edward T. Ordman

4. LINEARITY AND METRIZABILITY.

4.1. DEFINITIONS AND REMARKS. We now introduce the notions of linear limit and topological spaces and largely restrict consideration to such spaces. We shall introduce one metrizability standard, quoting some essential steps of the proof from Kelly since they are not especially relevant to our approach here.

In the following, X will ordinarily be a linear space (vector space) over a scalar field R (usually the real or complex numbers). Limit structures (or topologies may be assigned to X and to R, thus leading to limit structures or topologies on $X \times X$ and $R \times X$. The space (X,C;R,D) (usually denoted (X,C) or simply X) will be called a *linear space with limit structure* if there is no essential connection between the linear and limit structures. It will be called a *linear limit* (topological, etc.) *space* if and only if in addition to having a limit (topological, etc.) structure it has the property that $F(x,y) = x + y$ and $G(a,x) = ax$ are continuous functions from $X \times X$ and $R \times X$ respectively into (X,C).

4.2. THEOREM. *Let* (X,C;R,D) *be a linear space with limit structure. It is a linear limit space if and only if* $a_e \to a(D)$, $x_f \to x(C)$, *and* $y_g \to y(C)$ *imply that* $a_e x_f + y_g \to ax + y(C)$.

(Recall from 1.1 the convention on multiple subscripts in a net.)

If $a_e \to a$ and $x_f \to x$, $(a_e, x_f) \to (a,x)$ in $R \times X$ and by continuity of $G: R \times X \to X$, $a_e x_f \to ax$. Similarly for $x + y$.

To show F continuous, let $(x_f, y_g) \to xy$ in $X \times X$. By projection, $x_f \to x$ and $y_f \to y$. By hypothesis, $x_f + y_g \to x + y$; hence F is continous. G is similar.

4.3. THEOREM. *Let* (X,C;R,D) *be a linear space with a first countable* $\mathcal{T}_1$ *structure (i.e. such that* X *and* R *are first-countable* $\mathcal{T}_1$*-spaces). Then* X *is a linear* $\mathcal{T}_1$*-space if for all sequences* $a_n \to a$, $x_n \to x$, *and* $y_n \to y$, *we have that* $a_n x_n + y_n \to ax + y(C)$.

The condition is clearly implied by that for nets above; we must show that in a first countable $\mathcal{T}_1$-space it is equivalent. Let $a_e \to a(D)$, and so on, and

[1]This is part 2 of a two-part paper. Part 1 appeared in Volume 1, Number 2, (September, 1969), pp. 79-96.

examine $a_e x_f + y_g$. For every subnet $a_{e(d)}x_{f(d)} + y_{g(d)}$ we have $a_{e(d)} \to a$, $x_{f(d)} \to x$, and $y_{g(d)} \to y$, each through a system of neighborhoods which has a countable base. Suppose the bases to be enumerated; then we may find a d(n) such that $a_n = a_{e(d(n))}$ is in the nth neighborhood of a, x_n in the nth neighborhood of x, y_n in the nth neighborhood of y. Hence a_n converges to a and similarly for x and y, so by hypothesis $a_n x_n + y_n$ converges to ax + y. It follows (even though $a_{e(d(n))}$ is not a subnet of $a_{e(d)}$, only a sequence with smaller range) that $a_{e(d)}x_{f(d)} + y_{g(d)}$ is at least once in each neighborhood of ax + y. But if $a_e x_f + y_g$ were frequently outside any neighborhood, a subnet could be found which would be always outside; thus $a_e x_f + y_g$ must fail to be frequently outside any neighborhood, that is, it must eventually be in each neighborhood and converge. We have now established the hypothesis of 4.2, so the theorem follows.

4.4. REMARKS. It is clear that if the initial spaces are topological, the product limit spaces have the product topology, continuity reduces to topological continuity, and the linear space is a linear topological space. However, a much stronger result holds. We shall show in 4.9 that whenever X is a linear $\mathcal{T}_1$-space then X is automatically a linear topological space. This will be done through the introduction of uniform spaces, which are useful in giving a metrizability standard: a first countable uniform space is metrizable. Notions of bounded sets and local boundedness are introduced since they provide a net-definable sufficient condition for metrizability and necessary condition for normability. Prior to developing this machinery, we introduce a simple example.

4.5. EXAMPLE. Convergence pointwise. Let (X,p) be the space of functions from the reals (or [0,1] if you prefer) into the reals with convergence pointwise: f_d converges to f (pointwise) if and only if for every (x,r) where x is real and r is positive, eventually $|f(x) - f_d(x)| < r$. This is thus clearly a $\mathcal{T}_1$-sapce; an argument that it is topological is omitted and will in fact follow from later results. Linearity of convergence (4.2) holds since it holds pointwise. The space is, however, not first countable. Let g be a function with value 1 on the rationals, 0 on the irrationals. It is everywhere discontinuous and thus not a limit of a sequence of continuous functions (see e.g., Boas). On the other hand, it is in the closure of the set of all continuous functions. Let D be the collection of finite subsets S of the reals, and f_S equal g on the set S and anything continuos elsewhere (for instance, polygonal). Then $\{f_S, D, \supset\} \to g$ (pointwise). It follows that (X,p) is not a sequential space, and thus not first countable.

4.6. DEFINITION. A *base for a uniformity* for a set X is nonvoid collection $\mathcal{U}$ of subsets of $X \times X$ such that:
(a) each U in $\mathcal{U}$ contains (x,x) for all x in X, (b) if U is in $\mathcal{U}$, $\mathcal{U}$ contains a subset of $U^{-1} = \{(x,y): (y,x) \in U\}$, (c) for U there is some V in $\mathcal{U}$ such that $V \circ V = \{(x,z): \text{there exists } y \text{ with } (y,z),(x,y) \in V\} \subset U$, and (d) the intersection of two members of $\mathcal{U}$ contains a member of $\mathcal{U}$.

4.7. SUMMARY OF STANDARD RESULTS. We shall use without proof many properties of uniform spaces, readily available in, e.g., Kelly or Bourbaki. For U in $\mathcal{U}$, U(x) is the collection of y such that (x,y) is in U; U(x) contains x and may be regarded as a neighborhood. The resultant space is a topological space due to (c) above; a set is *open* if and only if it contains some U(x) for each x in it. This topology is called the uniform topology. A *uniform topology* is T_2 if it is T_1, which is equivalent to requiring the intersection of all sets in $\mathcal{U}$ to be exactly the diagonal (the collection of pairs (x,x)). The space is first countable if $\mathcal{U}$ may be chosen to be countable. We will accept the theorem (6.13, Kelley, *General Topology*) that any first-countable uniform space is pseudo-metrizable (metrizable if T_1). The hard work having been avoided, the following is proven largely to illustrate these concepts. We then go on to show that a linear $\mathcal{T}_1$-space is uniform.

4.8. THEOREM. *Any metric space is uniformizable and first-countable.*

Clearly open neighborhoods are provided by open spheres; we find a countable basis by picking spheres with rational radii (or simply spheres of radius 1/n for each n). To find a basis for a uniformity, Let U_r be the set of pairs (x,y) such that $d(x,y) < r$ and let $\mathcal{U}$ be the collection of U_r for, say, rational r. (a), (b), and (d) of 4.6 are apparent; (c) follows from the triangle inequality upon letting the "radius" r of V be half that of U. Finally, it is clear that the notion of convergence (in this case, the topology) generated by the uniform structure is the same as that generated by the metric. We remark also that the space is T_2 since $d(x,y) = 0$ implies $x = y$; if x and y were distinct and $d(x,y) = 0$, (x,y) would be in every U_r and y in each $U_r(x)$.

4.9. THEOREM. *Let* X *be a linear* $\mathcal{T}_1$*-space. Then it is uniform (hence topological by* 4.7 *and thus a linear topological space).*

For each neighborhood U(0) we define the set $U^* = \{(x,y): y - x \in U\}$, and let $\mathcal{U}$ be the collection of sets U^* for all U in some base for a neighborhood system at 0. We now establish that $\mathcal{U}$ is a base for a uniformity.

(a) 0 is in U, so (x,x) is in U*; (b) scalar multiplication is continuous, in particular f(x) = -x is continuous and for each U there is a neighborhood V with $-1 \cdot V \subset U$, i.e. $-1 \cdot V^* \subset U^*$; (c) f(a,b) = a + b is continuous on $X \times X$; f(0,0) equals 0. By 3.5 there is a neighborhood U_1 of (0,0) such that $f(U_1) \subset U$. Call a basis neighborhood contained in U_1, $V_1 \times V_2$, and let $V = V_1 \cap V_2$. Then V + V is contained in U. Suppose (x,y), (y,z) are in V*; (y - x) and (z - y) are in V, their sum (z - x) in U, and (x,z) in U*; (d) let U* and V* be given; let W be a subset of $U \cap V$ be another neighborhood of 0. If (x,y) is in W*, y - x is in W, U and V, so (x,y) is in both U* and V*. We must now show that convergence in the uniform topology is the same as the original convergence; this is immediately true at 0 since the neighborhoods are the same, and true at other points by translation since the space is linear.

For clarification, we now prove that this space actually is topological. Let U be a neighborhood of 0 and let N be the collection of x such that some $V^*(x) \subset U$; N contains some neighborhood of each of its points and is thus an open neighborhood of 0. To see this let x be in N, $V^*(x) \subset U$, and $W^* \circ W^* \subset V^*$. For every y in $W^*(x)$, $W^*(y) \subset W^*(W^*(x)) \subset V^*(x) \subset U$ so y is in N; thus N contains a neighborhood $W^*(x)$ for each x in N. The same argument holds about points other than 0, so the open neighborhoods form a base and the space is topological by 2.12.

4.10. EXAMPLE. This proves that pointwise convergence (4.5), since linear $\mathcal{T}_1$, is linear topological. However, since it is not first countable, it is not metrizable.

4.11. DEFINITIONS. Let (X,C;R,D) be a linear space with limit structure, R the real or complex numbers. A subset A of X is *bounded* if for each sequence $\{x_n\}$ in A and each sequence $a_n \to 0$ of real positive scalars, $a_n x_n \to 0(C)$. In particular, if X is a linear limit space, each singleton is bounded. ($a_n x \to 0x = 0$.)

Now let X be a linear limit space. It is *locally bounded* if and only if each convergent net is eventually in some bounded set. If there is a bounded open set, some translate of it covers every point so the space is locally bounded (similarly if some point has a bounded neighborhood). This proves also the first half of the following Theorem.

4.12. THEOREM. *A linear $\mathcal{T}$ space is locally bounded if and only if each pseudo-neighborhood system contains a bounded pseudo-neighborhood.*

"If" is immediate. If some point x has a pseudo-neighborhood system containing no bounded set, the net having that system as its associated filter (1.3c) would converge to x but fail to be in any bounded set, contradicting local boundedness.

4.13. COROLLARY. *A linear topological space is locally bounded if and only if there is a bounded open set.*

Zero has a bounded neighborhood which has an open subneighborhood.

4.14. THEOREM. *Let X be a linear space with $\mathcal{T}_1$ structure. A subset A of X is bounded if and only if each neighborhood U of 0 contains some (1/n)A.*

Suppose some U fails to contain such a set; each (1/n)A contains a point $(1/n)x_n$ not in U. x_n is a sequence in A, $1/n \to 0$, but x_n/n remains outside U and cannot converge to 0, so A is unbounded. Conversely, if $a_n \to 0$, a_nA is eventually in each U so for any sequence x_n in A, $a_nx_n \to 0(C)$.

4.15. THEOREM. *Let X be a linear topological space and be locally bounded. Then it is first countable and hence metrizable.*

By 4.12 we may choose a bounded neighborhood of 0, say U. By 4.14 each neighborhood contains sone (1/n)U; we must show each (1/n)U contains a neighborhood V of 0. $f(x) = nx$ is a continuous function and $f(0) = 0$ so by 3.5 there is a neighborhood V for each n such that $nV \subset U$, i.e., $V \subset (1/n)U$. Thus the sets (1/n)U form a countable base for the neighborhood system.

4.16. EXAMPLE. Convergence in measure. Suppose we let X be the space of measurable functions from the interval [0,1] into the reals with functions equal a.e. identified. Now for each function f in X and positive rational number r, define $N_r(f) = \{g: \mu\{x: |f(x) - g(x)| > r\} < r\}$. We say that a net f_d converges to f (measure) if and only if f_d is eventually in each $N_r(f)$. Clearly (X,measure) is a first-countable $\mathcal{T}_1$-space; we shall now show that it is a linear limit space and thus metrizable. First, suppose $f_n \to f$ and $g_n \to g$; $|f_n(x) + g_n(x) - f(x) - g(x)|$ is less than $|f_n(x) - f(x)| + |g_n(x) - g(x)|$ so $f_n + g_n$ converges to $f + g$. If $f_n \to 0$, clearly $af_n \to 0$ for any constant a; if $a_n \to 0$ and $f_n \to 0$, $a_n \leqq 1$ eventually and clearly $a_nf_n \to 0$. If $a_n \to 0$ and f is some element of X, $a_nf \to 0$ since $a_nf \to 0$ (pointwise).

(Note that it is essential that X consist of functions defined on a <u>bounded</u> interval. The real-real function $f(x) = x$ satisfies $f(x)/n$ not in $N_1(x)$ for <u>any</u> n and cannot belong to any linear limit space of convergence in measure. Equivalently, {f} is unbounded which is impossible in a linear limit space.)

We now have that if $a_n \to a$, $f_n \to f$, then $a_nf_n - af = (f_n - f)(a_n - a) + a_n(f_n - f) + f(a_n - a) \to 0$. Thus by Theorem 4.3, (X,measure) is a first-countable $\mathcal{T}_1$-space, and hence topological and metrizable.

It is of interest that (X,measure) is not locally bounded. Suppose we consider the set of ordered pairs (m,n) ordered by first elements, and set $f_{(m,n)}(x)$

equal to n for x in [0,1/m] and zero elsewhere. The net f eventually differs from zero only on an arbitrarily small interval, so it converges to zero. On the other hand, any set which $f_{(m,n)}$ is eventually in contains all points of the form $f_{(m,n)}$ for some m. While $1/n \to 0$, $(1/n)f_{(m,n)} = f_{(m,1)} \neq 0$ and does not converge to zero, so by definition 4.11 (X,measure) is not locally bounded. This assures us that there is no bounded open set without even requiring us to explicitly determine the topology.

4.17. EXAMPLES. Convergence pointwise and almost everywhere. In 2.18 we defined the linear $\mathcal{T}_1$-space of almost everywhere convergence, (X,a.e.), and in 4.5 the linear topological space of convergence pointwise (X,p.). We noted in 4.5 that (X,p.) was not first-countable, so it is not locally bounded and not metrizable; it is clear that the same results hold for (X,a.e.) (it is obviously nonmetrizable, since nontopological, and the example of 4.5 may be modified to show non-first countability). The net $f_{(m,n)}$ also shows non-local boundedness for (X,a.e.) and for (X,p.) (with modification to let each $f_{(m,n)}(0)$ be 0).

5. LOCAL CONVEXITY AND NORMABILITY.

The first part of this section discusses local convexity, a property often desired in linear spaces but apparently difficult to handle in terms of convergence. The second part gives Kolmogorov's normability standard, as a logical followup to convexity and boundedness notions; much of it is taken from Taylor, 3.3 - 4.1.

5.1. DEFINITIONS. In the following X will be a linear space over the real or complex numbers. The *segment* between points a and b of X is the collection of points of form na + mb, where n and m are non-negative and sum to 1. A set is *convex* if and only if it contains the segment between any two of its points; clearly an arbitrary intersection of convex sets is convex. The *convex hull* A_c of a subset A of X is the smallest convex set containing A, that is, the intersection of all convex sets containing A.

5.2. LEMMA. *Let* T *be the collection of points* $x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ *where* n *is arbitrary and finite, each* x_k *is in* A, *each* a_k *is nonnegative, and the* a_k *sum to* 1. *Then* T *is the convex hull of* A.

T is clearly convex and contains A; we must show any convex superset of A contains T. The superset contains points of form $a_1x_1 + a_2x_2$, and thus points of form $b_1(a_1x_1 + a_2x_2) + b_2x_3$, where $b_1 + b_2 = 1$ and hence $b_1a_1 + b_1a_2 + b_2 = 1$.

By induction on n, the superset contains each point of T.

5.3. THEOREM. *If X is a linear limit space and A is open, A_c is open.*

If x is in A_c it is of form $a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where a_1 is non-zero. By linearity $a_1A + a_2x_2 + \cdots + a_nx_n$ is an open set, containing x and contained in A_c. Thus A_c contains an open set about each of its points, and is open.

5.4. LEMMA. *If* A *is convex and* a *and* b *are real and positive, then* $aA + bA = (a + b)A$.

Clearly $(a + b)A$ is a subset of $aA + bA$. If x_1, x_2 are in A and $y = ax_1 + bx_2$, $y/(a + b) = ax_1/(a + b) + bx_2/(a + b)$ which is in A; thus y is in $(a + b)A$.

5.5. DEFINITIONS. Just as local boundedness requires that "neighborhoods eventually be bounded," local convexity will require neighborhoods to be convex. While nets do not lend themselves easily to convexity restrictions, the following rather artificial construction will suffice for our purposes. A net will be called *convex* if each basis set of its associated filter is convex; we wish to define a sort of "convex hull" for nets. Suppose f is a net on domain D; we will construct a *convex hull* g as follows: Let E be the class of triples (d,P,Q) where d is in D, P is an n-tuple (for some finite n) of points of D all of which follow d, and Q is an n-tuple of nonnegative fractions whose sum is one. Now let the points of E be ordered by their first elements, and let $g(d,P,Q) = a_1f_1 + a_2f_2 + \cdots + a_nf_n$ where a_k, f_k are the kth elements of Q and P respectively. Then g is a net whose associated filter base consists of convex sets, the convex hulls of the sets in the filter generated by f.

We now define a linear space with limit structure to be *locally convex* if and only if whenever any net f converges, its convex hull g also converges. This definition is actually a bit bulky for showing that a space is locally convex; it is quite useful in establishing counterexamples.

5.6. THEOREM. *Let* X *be a linear* $\mathcal{T}$ *space. Then* X *is locally convex if and only if each pseudo-neighborhood system contains a (coarser) pseudo-neighborhood system which has a basis consisting of convex sets.*

Suppose first the second condition holds and f converges to x, through a pseudo-neighborhood system $\mathcal{S}$. There is a system $\mathcal{S}'$, a subset of $\mathcal{S}$, which has a basis consisting of convex sets. f is eventually in each set of that basis (since it is eventually in each set of $\mathcal{S}$ and each basis set is in $\mathcal{S}$) and thus g is eventually in each set of the basis and must converge. Thus X is locally convex.

Conversely, suppose the convergence of any f implies the convergence of its convex hull g, and let $\mathcal{S}$ be any pseudo-neighborhood system. Then the net whose associated filter in $\mathcal{S}$ converges, so its convex hull g converges. As noted in 5.5, g has an associated filter base consisting of convex sets; call the filter so generated $\mathcal{S}^*$. Any net eventually in each set of $\mathcal{S}^*$ is in every set that g is, and thus converges; that is, $\mathcal{S}^*$ is a pseudo-neighborhood system with a basis of convex sets. Finally, each set in $\mathcal{S}^*$ contains a set of $\mathcal{S}$ so $\mathcal{S}^*$ is coarser than $\mathcal{S}$.

5.7. COROLLARY. *Let* X *be a linear topological space. Then* X *is locally convex if and only if each neighborhood contains a convex neighborhood.*

In 5.6 there is now only one neighborhood system; it has a basis of convex sets.

5.8. THEOREM. *Let* X *be a linear space with a locally convex* $\mathcal{L}$ *structure (that is, it need not be a linear limit space). Then the convex hull* B_c *of any bounded set* B *is also bounded.*

We rely largely on the definition of "bounded"; a sequence in B_c will be chosen, multiplied by a sequence $a_n > 0$ of scalars, and the resulting sequence shown to be a subnet of the convex hull of a sequence constructed from points of B. Suppose y_n to be a sequence in B_c, and let us examine $a_n y_n$. For each n there is an integer k(n), a k(n)-tuple $Q_n = (r_1, r_2, \cdots, r_{k(n)})_n$ of nonnegative reals summing to one, and a k(n)-tuple $S_n = (x_1, x_2, \cdots, x_{k(n)})_n$ of points of B such that $y_n = (\Sigma_{i=1}^{k(n)} r_i x_i)_n = \Sigma_{i=1}^{k(n)} r_{i,n} x_{i,n}$, $a_n y_n = \Sigma\, r_{i,n} a_n x_{i,n}$. Let us now change notation, letting $b_{j,n} = a_n$ for j = 1, 2, $\cdots$, k(n) and then letting $c_1 = b_{1,1}$, $c_2 = b_{2,1}$, $\cdots$, $c_{k(1)} = b_{k(1)}$, $c_{k(1)+1} = b_{2,1}$, $\cdots$ and so on, effectively putting the a_n in a new sequence where a_n is repeated k(n) times. Now let P_n be the k(n)-tuple $(z_{m+1}, z_{m+2}, \cdots, z_{m+k(n)})$ where c_{m+1} is $b_{1,n}$ and thus $c_{m+k(n)} = b_{k(n),n}$, and the z's are to be defined. The sequence of scalars c_n converges to zero since a_n does. Choose a sequence z_n of points of B by letting $z_m = x_{i,n}$ where $c_m = b_{1,n}$, i.e., order the x's in the same way as the b's. Now by the boundedness of B, the sequence $c_n z_n$ converges to zero and so does its convex hull g(n,P,Q) (where notation is as in 5.5). Let $h_n = g(n, P_n, Q_n)$; h is then a subset of g and since defined on the natural numbers it is a sequence; since X is an $\mathcal{L}$ space, h converges to zero. However, $h_n = g(n, P_n, Q_n) = \Sigma_{i=1}^{k(n)} r_{i,n} c_{m+i} z_{m+i} = \Sigma\, r_{i,n} b_{i,n} x_{i,n} = \Sigma\, r_{i,n} a_n x_{i,n} = a_n y_n$. Thus we see that $a_n y_n$ converges to zero, and B_c is bounded.

The complexity of this proof is due to the lack of strong restrictions on the space X; in a locally convex linear topological space, the theorem is almost immediate from 4.14 and 5.7. This proof has been included primarily to show that local convexity may imply nontrivial results even in spaces so weak that they have no "neighborhoods" (or even pseudo-neighborhoods) to be convex. That the result is actually non-trivial is shown by the first example of 5.9.

5.9. EXAMPLES. (a) We shall give a space X which is linear, has an unrelated $\mathcal{T}_1$ structure, but fails to be locally convex, which has a bounded set whose convex hull is unbounded. Let X be the complex plane, nets converge to non-zero points if and only if eventually constant, and let a net converge to zero if and only if (1) it converges in the Euclidean metric and (2) it is eventually in the axes (i.e. beyond some point each term is either real or imaginary, although terms of both sorts are allowed.) This is clearly a $\mathcal{T}_1$ convergence and fails to be locally convex (since no neighborhood of zero is convex). Now if B is the set consisting of the points 1 and i, it is clear that B is bounded; but its convex hull contains the point $1/2(1 + i)$, which yields a nonconvergent sequence when multiplied into any positive real sequence $a \to 0$.

(b) Convergence pointwise and almost everywhere. These convergences are enough related that we need prove local convexity only for the first. It is clear that the set of functions which differ from f by less than r at the point x is convex; thus the neighborhoods (which are finite intersections of such sets for varying x and r) are convex and by 5.7 the space is locally convex.

(c) Convergence in measure. Convergence in measure, shown in 4.16 not to be locally bounded, is also not locally convex. Suppose D to be the class of ordered pairs (n,m) with n natural and m an integer from 1 to n, with D ordered by first elements. Let $f_{(n,m)}$ be n on the interval $[m - 1/n, m/n]$ and zero elsewhere in [0,1]. Clearly $f_{(n,m)} \to 0$ (measure). Now let us consider the convex hull of the net f. No matter how far out we go, the net still contains all of the functions $f_{(n,m)}$ for some n, and hence their "average" $f_{(n,1)}/n + f_{(n,2)}/n + \cdots + f_{(n,n)}/n = f_{(1,1)}$. Since the convex hull frequently has the value $f_{(1,1)}$ it clearly cannot converge to 0, and (X,measure) is not locally convex.

5.10. DEFINITIONS. We now turn to the problem of whether the topology of a linear topological space is equivalent to that generated by a norm. A *norm* $||x||$ on a space X is a function from the linear space into the non-negative reals such that (a) $||x + y|| \leq ||x|| + ||y||$; (b) if a is a scalar (i.e. real or complex) $||ax|| = |a|\cdot||x||$; (c) if $||x|| = 0$, then $x = 0$. A norm which fails to satisfy (c) is a *pseudo-norm*.

A *balanced* (equilibre, star-like, etc.) set is a set A such that if $|a| \leq 1$ then $aA \subset A$. Notice that a set may be balanced over the real numbers but fail to be balanced over the complex numbers, so in some cases the scalar field must be carefully noted. The *balanced hull* A_b of A is the intersection of all balanced sets containing A; then $A_b = \cup aA$ for $|a| \leq 1$.

5.11. THEOREM. *(a) Every normed space is metrizable. (b) Every normed space is locally convex and locally bounded.*

(a) In a normed space, the distance between any two points x and y is $||x - y||$; this is clearly a metric and is the ordinary way of defining the topology in a normed space. (b) It follows from 5.10(a) and (b) that the unit sphere is convex and bounded; hence a small scalar multiple of it (equivalently, a small sphere) is a convex subneighborhood of each neighborhood of zero (and by translation, of any other point). (In fact, a space is locally convex and locally bounded if and only if there is a bounded open convex set.)

5.12. LEMMA. *If* A *is balanced then the convex hull* A_c *of* A *is balanced.*

If x is in A_c, $x = \Sigma a_i x_i$ as in 5.3. For $|a| \leq 1$, $ax = \Sigma a_i(ax_i)$ which is in A_c since each ax_i is in A.

5.13. THEOREM. *Let* U *be a convex neighborhood of* 0 *in the linear topological space* X*; then* U *contains a convex balanced neighborhood of* 0.

Let $U_1 = U \cap -1\cdot U$ and $U_2 = 1/2\cdot U_1$ and thus U_1 and U_2 are convex and in fact balanced with respect to the reals; we proceed to the more difficult complex case. By continuity of multiplication by i we may find an open neighborhood V_1 of zero, smaller than U_2, such that $i\cdot V_1$ and thus $V_1 \cup i\cdot V_1 \subset U_2$. Let V be the convex hull of the balanced hull of V_1; by 5.3, 5.12, and the fact that the balanced hull is a union of scalar multiple of the original (open) set, V is convex, balanced, and open; we must show $V \subset U$. Let x be in the balanced hull of V_1; then for some a + bi of absolute value less than or equal to 1, x is in $(a + bi)V_1 \subset aV_1 + biV_1 \subset 1/2aU_1 + 1/2bU_1 = (a + b)(U_1/2)$. But $a^2 + b^2$ is less than or equal 1, so $|a + b| < 2$ and x is in U_1 and thus in U. Since the balanced hull of V_1 is a subset of the convex set U, V must also be a subset.

5.14. DEFINITIONS. A set K is *absorbing* if 0 is in K and for each x in X there is a positive real r such that $|a| \geq r$ implies x in aK. It has been remarked that any neighborhood of zero is absorbing (this is equivalent to the assertion that each singleton is bounded). Any convex absorbing set may be used as an "approximate unit sphere" to define an "approximate norm": the *Minkowski functional* p(x) of an absorbing set K is $p(x) = \{\inf a: a > 0, x \text{ in } aK\}$.

5.15. THEOREM. *Let* K *be convex and absorbing. Then: (a)* $p(0) = 0$, *(b)* $p(x + y) \leqq p(x) + p(y)$, *(c) if* K *is balanced (real or complex) then* $p(ax) = |a|p(x)$ *(for real or complex* a, *respectively), (d) if* K *is open,* $K = \{x: p(x) < 1\}$, *and (e) if* K *is bounded and* X *a* T_1 *space,* $p(x) = 0$ *implies* $x = 0$.

(a) 0 is in aK for every a; (b) let x be in aK, y in bK; x + y is then in $aK + bK = (a + b)K$, so $p(x + y) \leqq a + b$ for every such a and b and thus we have $p(x + y) \leqq p(x) + p(y)$; (c) if $a = 0$, apply (a). Otherwise, if x is in bK then $(a/|a|b)x$ is in K, hence ax is in $|a|bK$, $p(ax) \leqq |a|b$ and thus $p(ax) \leqq |a|p(x)$. Substituting a'x' for x and 1/a' for a, $p(a'x') \geqq |a'|p(x')$; (d) clearly if $p(x) < 1$, x is in $1 \cdot K = K$, so x is in $a^{-1}K$ and $p(x) \leqq a^{-1} < 1$; (e) if x is not 0 there is a balanced neighborhood U of 0 not containing x. Since K is bounded there is a scalar $b > 0$ with $bK \subset U$; if x is in $aK \subset (a/b)U$ then $a/b > 1$, $a > b$, so $p(x) \geqq b > 0$.

5.16. THEOREM. *A linear topological space is pseudo-normable if there is a bounded convex neighborhood of 0; normable if also* T_1.

Let U be the given neighborhood; there is an open subneighborhood K that is also balanced and, since a neighborhood, absorbing. 5.15(a), (b), (c), (e) show that p is a pseudo-norm. The topology generated by this norm is the same as the original since K is a bounded open set in each topology 5.15(d) and hence the sets $(1/n)K$ form a base for the neighborhood system at 0 in each topology.

5.17. EXAMPLE. The simpliest normed space commonly referred to is the space of bounded real-real functions, with $||f|| = \sup|f|$. It is clear that the resultant convergence is exactly uniform convergence of functions.

5.18. DEFINITIONS. We now turn briefly to another aspect of local convexity which may be of interest. Let X be a linear space over the real or complex numbers: the *convex core* convergence (c) is determined by letting $f_d \to f(c)$ if and only if $f_d - f$ is eventually in each absorbing, balanced, convex set.

5.19. THEOREM. (X,c) *is a locally convex linear topological space.*

It is clear that (X,c) is a $\mathcal{T}_1$ space; denote the collection of absorbing balanced convex sets, the neighborhood basis at 0, by $\mathcal{B}$. Clearly any nonzero scalar multiple of a set of $\mathcal{B}$ is again in $\mathcal{B}$, and each U in $\mathcal{B}$ is convex so $(1/2)U + (1/2)U = U$ by 5.4. If f_d and g_e converge to 0 each is eventually in $(1/2)U$ and $f_d + g_e \to 0(c)$. For multiplication, suppose $a_d \to 0$ and $f_e \to 0(c)$. Then $a_d f \to 0$ since each U is absorbing; $af_e \to 0$ since f_e is eventually in $(1/a)U$. Eventually $|a_d| \leqq 1$ and since each U is balanced $a_d f_e \to 0$. Thus if a_d and f_e converge to a and f respectively, $a_d f_e - af = (a_d - a)(f_e - f) + a(f_e - f) +$

$f(a_d - a) \to 0(c)$. Translation of limits is trivial; thus by 4.2 and 4.9, (X,c) is a linear topological space (this structure is also called the *convex core topology* for X). Also, (X,c) is locally convex because each neighborhood contains a convex neighborhood.

5.20. THEOREM. *Let* (X,c) *be the convex core convergence on* X *and* d *another convergence for which* (X,d) *is a locally convex linear topological space. Then* c *is finer than* d, *i.e., has fewer convergent nets* (see 3.1, 3.2).

Let A be open in (X,d) with 0 in A. Then there is a convex N open in (X,d) with 0 in $N \subset A$, and an open $U \subset N$ which is convex, balanced, and absorbing; thus U contains a neighborhood of 0 from (X,c). Hence any net which converges in (X,c) to 0 must also converge in (X,d) and the result follows similarly for other points.

BIBLIOGRAPHY

See part 1 of this paper, Vol. 1, #2, pp. 95-96.

Edward T. Ordman (Senior)
Kenyon College (1964)