JCMCC

THE JOURNAL

of

COMBINATORIAL MATHEMATICS

and

COMBINATORIAL COMPUTING

VOLUME 2

OCTOBER, 1987

MENGER'S THEOREM AND SHORT PATHS

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ABSTRACT

For positive integers d and m, let $P_{d,m}(G)$ denote the property that between each pair of vertices of the graph G, there are m vertex disjoint (except for the endvertices) paths each of length at most d. Minimal conditions involving various combinations of the connectivity, minimal degree, edge density, and size of a graph G to insure that $P_{d,m}(G)$ is satisfied are investigated. For example, if a graph G of order n has connectivity exceeding $(n - m)/d + m - 1$, then $P_{d,m}(G)$ is satisfied. This result is the best possible in that there is a graph which has connectivity $(n - m)/d + m - 1$ that does not satisfy $P_{d,m}(G)$. Also, if an m-connected graph G of order n has minimal degree at least $\lfloor (n - m + 2)/\lfloor (d + 4)/3 \rfloor \rfloor + m - 2$, then G satisfies $P_{d,m}(G)$. Examples are given that show that this minimum degree requirement has the correct order of magnitude, and cannot be substantially weakened without losing property $P_{d,m}$.

* Research partially supported under NSF grant no. DCR-8503922
** Research partially supported under NSF grant no. DMS-8603717
*** Research partially supported under ORN grant no. 00014-85-K-0694
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JCMCC 2(1987), pp. 235-253

1. INTRODUCTION

Consider a graph G which will represent a computer network with each vertex representing a processor in the network and each edge representing a two-way communication link. To insure that the network is fault-tolerant with respect to processor failures, it is necessary that the number of vertex disjoint paths between each pair of vertices exceed the number of possible failures. In particular, if there is a possibility of f processor failures, then it is crucial that there be at least $f + 1$ independent paths between each pair of vertices to insure that at least one message arrives. Also, processors could have "byzantine" failures: instead of failing to forward the message they change the message. To be able to insure that the majority message received was the original message transmitted in a network with f such failures, the number of vertex disjoint paths between pairs of vertices must be greater than twice the number of possible failures (i.e. $2f + 1$). Connectivity is clearly the crucial graph concept in both cases. However, the length of time for the messages to arrive is also important, so it is desirable that the vertex disjoint paths be short. This transalates into the fact that between each pair of the vertices of the graph G there is a specified number of paths, each with a bounded number of vertices.

For positive integers d and m, let $P_{d,m}(G)$ denote the property that between each pair of vertices of the graph G, there are at least m vertex disjoint paths each of length at most d. The graph G representing a computer network prone to processor failures should satisfy $P_{d,m}(G)$ for appropriate values of d and m. This is one motiviation for studying property $P_{d,m}$.

Menger's classical result [8] on connectivity solves the problem of the existence of a system of such paths, if there is no concern for the length of the paths in the system. Although Menger's theorem gives no information about the length of the paths, the "length problem" has been studied. For example, Bond and Peyrat studied the effect of adding or deleting edges on the diameter of a network in [2], and Chung and Garey considered diameter bounds for altered graphs in [3]. Menger type results for paths of bounded length were proved by Lovász, Neumann-Lara, and Plummer in [7] and by Pyber and Tuza in [11], and Mengerian theorems for long paths were given by Montejanao and Neumann-Lara in [9] and by Hager in [6]. In [10] property $P_{d,m}$ and its application to computer networks and distributed processing was introduced.

We will investigate conditions on various graphical parameters of a graph G that are sufficient to insure that $P_{d,m}(G)$ is satisfied. Various combinations of size, connectivity, minimum degree, and density properties that imply $P_{d,m}(G)$ will be considered. In particular, the following four results will be proved.

The first result does not have any connectivity assumptions, and thus, has the extremely strong size and degree conditions needed to insure that a graph is m-connected. The proof is straightforward.

THEOREM 1: Let $d \geq 2$ and m be positive integers. Let G be a graph of order n, for which at least one the following conditions hold:

(1) G has at least $\binom{n-1}{2} + m$ edges.

(2) G has minimum degree $\lfloor (n + m)/2 \rfloor$.

Then G satisfies $P_{d,m}$. Further, each condition is sharp.

Sufficiently large connectivity will guarantee that $P_{d,m}(G)$ is satisfied, and the following result gives the minimum such connectivity that will suffice.

THEOREM 2: Let d and m be positive integers, and let G be a graph of order n. If G has connectivity exceeding $(n - m)/d + m - 1$, then $P_{d,m}(G)$ is satisfied. This result is the best possible in that there is a graph which has connectivity $(n - m)/d + m - 1$ that does not satisfy $P_{d,m}(G)$.

If the graph is known to be m-connected, then the degree condition needed to insure $P_{d,m}(G)$ is greatly reduced, as the following result indicates.

THEOREM 3: Let m and d be positive integers, and let G be an m-connected graph of order n. If G has minimum degree exceeding $\lfloor (n - m + 2)/\lfloor (d + 4)/3 \rfloor \rfloor + m - 2$, then G satisfies $P_{d,m}(G)$.

If a graph G has connectivity at least m, then a density condition on the edges of G can insure that $P_{d,m}$ is satisfied. With this in mind, for a graph G and any positive integer s, let $\text{den}(G,s)$ be the minimum number of edges in any induced subgraph of G on s vertices. The following result gives a sufficient restriction on den(G,s) to insure $P_{d,m}$.

THEOREM 4: Let m and $d \geq 2$ be positive integers, and let G be a k-connected graph $(k \geq m)$ of order n. If there exists some s $(s \leq (k-m+1)\lfloor d/2 \rfloor)$, such that G satisfies the inequality

$$\text{den}(G,s) \geq (3/4)s\lceil s/\lfloor d/2 \rfloor \rceil,$$

then G has property $P_{d,m}(G)$.

The density condition in Theorem 4 is not the best possible, but an example will be given in the next section to indicate that it has the correct order of magnitude.

Notation and standard definitions in the paper will generally follow that found in [1]. Any special notation will be described as needed.

2. EXAMPLES

Graphs with the same connectivity, and even the same minimal degree, can have vastly different Menger type path systems. Consider the generalized wheel graph $W_{m-2,n-m+2} = K_{m-2} + C_{n-m+2}$, which has order n, and connectivity and minimal degree m. It is easy to see that this graph does not satisfy $P_{n-m,m}$, since any m internally vertex disjoint paths between a pair of adjacent vertices on the rim of the wheel will have one path that contains all of the vertices of the rim of the wheel. On the other hand, the m-cube Q_m has order $n = 2^m$, connectivity and minimal degree m, and it satisfies $P_{m+1,m}$. This last assertion is easy to verify by an induction argument on the index m of the m-cube.

"Wheel type" graphs give important information on the extremal properties related to $P_{d,m}$. Consider the wheel graph $W_r = K_1 + C_r$ that has r spokes and r vertices on the rim. By replacing each vertex of W_r with a complete graph, and making each vertex of the corresponding complete graph adjacent to the vertices in the neighborhood of the replaced vertex, a new graph is obtained. A finite number of applications of this process gives a family of *generalized wheels*. Order the vertices of W_r starting with the center and followed by the vertices on the rim in a natural order around the cycle. For positive integers $p(i)$ $(0 \leq i \leq r)$, there is a generalized wheel, which we will denote by $W(p(0), p(1), \ldots, p(r))$, obtained from W_r by replacing the i^{th} vertex with a complete graph $K_{p(i)}$. This graph has $p(0) + \ldots + p(r)$ vertices. Note that if each $p(i) = 1$, the graph is isomorphic to W_r.

In many of the cases of interest to us, most of the $p(i)$'s in the generalized wheel will be the same, so we will adopt the more compact notation of representing the sequence $(p(j), \ldots, p(k))$ with $p = p(j) = \ldots = p(k)$ by $(k-j+1;p)$. Thus, $W(1,r;1) = W_r$ and $W(m-2,n-m+2;1) = K_{m-2} + C_{n-m+2}$, which is the generalized wheel considered earlier in this section. For the following families of generalized wheels, it will be assumed that $d \geq 2$ and m are fixed positive integers.

Select any integer n such that $n - m$ is divisible by d, and consider the generalized wheel

$$W(m-2,d;(n-m)/d,1,1).$$

Let x and y denote the two vertices on the rim of the generalized wheel that are associated with the complete graphs that contain a single vertex. This graph has order n, connectivity $m - 1 + (n - m)/d \geq m$, and $m - 1$ internally disjoint paths of length at most 2 between x and y. However, any path from x to y not using any of the $m - 2$ vertices in the center of the generalized wheel or the edge xy has length at least $d + 1$. Therefore, $W(m-2,d;(n-m)/d,1,1)$ does not satisfy $P_{d,m}$. We will verify in Theorem 2 that any graph with connectivity exceeding $(n - m)/d + m - 1$ does satisfy $P_{d,m}$.

Select any positive integer p, let $n = (d + 4)p + m - 4$, and consider the graph

$$W(m-2,3p-2,d-2;p,3p-2,1,1).$$

Again, let x and y be the vertices on the rim of the generalized wheel associated with the complete graphs with a single vertex. This graph has order n, minimum degree $3p + m - 3 = 3(n - m + 4)/(d + 4) + m - 3$, and connectivity $p + m - 1 \geq m$. Just as before, any path between x and y that does not contain the edge xy or any of the $m - 2$ vertices in the center of the graph has length at least $d + 1$. Thus, this graph does not satisfy $P_{d,m}$, but has minimum degree of the same order of magnitude as the degree condition in the hypothesis of Theorem 3.

Let $k \geq m$ be a fixed positive integer, and consider the following generalized wheel of order n which we will denote by H_1:

$$H_1 = W(m-2,d-1;k-m+1,n-m-(d-1)(k-m+1),1,1).$$

For n sufficiently large, this graph has connectivity k but does not satisfy $P_{d,m}$ for the same reasons described in the previous examples. Also, this graph is edge maximal in that the addition of any edge will insure that $P_{d,m}$ is satisfied.

Denote by H_2 the graph obtained from the graph

$$W(m-1,d-1;k-m+1,n-m-(d-1)(k-m+1)-1,1,1)$$

by deleting the edge between x and y, where x and y are the vertices on the rim associated with the complete graphs on a single one vertex. The graph H_2 also has order n, and does not satisfy $P_{d,n}$, but the addition of any edge will imply $P_{d,m}$.

Each of the graphs H_1 and H_2 are very dense graphs. One way to measure this density is to determine the number of edges contained in induced subgraphs. Recall, for each positive integer s and graph G, $den(G,s)$ is the minimum number of edges in any induced subgraph of G on s vertices. Thus, $den(H_i,s)$ is defined for $i = 1, 2$. It is easy to verify that $den(H_1,s) \geq den(H_2,s)$ for $s \leq n - m + 1$, and the reverse inequality is true for the remaining values of s. Let

$$den(n,k,s) = \min\{den(H_1,s),\ den(H_2,s)\}.$$

Theorem 4 gives some indication that there is a positive answer to the question: For a fixed s, must any k-connected graph G of order n with $den(G,s) > den(n,k,s)$ satisfy $P_{d,m}$.

3. RESULTS

There are obvious conditions on the size and minimum degree of a graph G that insure that $P_{d,m}(G)$ holds. For positive integers $r \le s$, consider the graph $K_{m-1} + (K_r \cup K_s)$, which has connectivity $m - 1$ and minimum degree $m + r - 2$. This graph, which we will denote by $G(r,s)$, clearly does not satisfy $P_{d,m}$, because of the connectivity condition. The graph $G(1,n-m)$ has $\binom{n-1}{2} + m - 1$ edges. We will verify in Theorem 1 that this is the maximum number of edges in a graph of order n that does not satisfy $P_{d,m}$. The graph $G(\lfloor (n-m+1)/2 \rfloor, \lceil (n-m+1)/2 \rceil)$ has minimal degree $\lfloor (n-m+1)/2 \rfloor$. It will be shown in Theorem 1 that this is the largest possible minimum degree in a graph not satisfying $P_{d,m}$.

THEOREM 1: Let $d \ge 2$ and m be positive integers. Let G be a graph of order n, for which at least one of the following conditions hold:

(1) G has at least $\binom{n-1}{2} + m$ edges.

(2) G has minimum degree $\lfloor (n + m)/2 \rfloor$.

Then G satisfies $P_{d,m}$. Further, each condition is sharp.

PROOF: Consider vertices x and y of G, and let N be those vertices of G which are adjacent to both x and y. If N has at least m vertices or just $m - 1$ vertices when x and y are adjacent, then $P_{d,m}$ is satisfied. In fact, all of the paths have length at most 2. Suppose this is not true. This implies G has at least $n - m$ edges, and either x or y has degree less than $\lfloor (n + m)/2 \rfloor$, a contradiction. The examples described prior to the statement of Theorem 1 imply the sharpness of the conditions, and completes the proof. ∎

Both of the conditions of Theorem 1 are so strong that there are m paths of length either 1 or 2 between each pair of vertices. Thus, the parameter d does not effect the size or minimum degree needed to imply $P_{d,m}$.

The generalized wheel $W(m-2,d;(n-m)/d,1,1)$ discussed in the previous section shows that in a graph G of order n, the connectivity of G must exceed $(n - m)/d + m - 1$ for $P_{d,m}$ to hold. Theorem 2 verifies that this is also sufficient.

THEOREM 2: Let m and d be positive integers, and let G be a graph of order n. If G has connectivity exceeding $(n - m)/d + m - 1$, then $P_{d,m}(G)$ is satisfied. This result is the best possible in that there is a graph which has connectivity $(n - m)/d + m - 1$ that does not satisfy $P_{d,m}(G)$.

PROOF: The necessity of the condition has already been exhibited. The sufficiency can be verified by a counting argument. Let x and y be vertices of G. Since G is k-connected for some $k > (n - m)/d + m - 1$, there are k internally vertex disjoint paths, say $P_1, \ldots, P_k$, between x and y. We can assume the paths are ordered by their length. If $P_{d,m}$ is not satisfied, then at most $m - 1$ of these paths have length at most d, so each P_i has length exceeding d for $i \geq m$. Also, the total number of internal vertices in the $m - 1$ shortest paths is at least $m - 2$, and there will be equality only when there is one path of length 1 and $m - 2$ paths of length 2. Therefore, by counting the internal vertices in the k paths along with x and y, we have the number of vertices in G is at least

$$d(k - m + 1) + (m - 2) + 2 > d(n - m)/d + m = n.$$

Since this is impossible, the proof of Theorem 2 is complete. ∎

The minimum degree in a graph G required to insure property $P_{d,m}(G)$ was considered in Theorem 1. Theorem 3 shows that the minimum degree needed is less if G is m-connected. However, one of the generalized wheel examples shows that the minimum degree must exceed

$$(3(n - m + 4)/(d + 4)) + m - 3.$$

Theorem 3 indicates that this is asymptotically the correct value.

THEOREM 3: Let m and d be positive integers, and let G be an m-connected graph of order n. If G has minimum degree exceeding $\lfloor (n - m + 2)/\lfloor (d + 4)/3 \rfloor \rfloor + m - 2$, then G satisfies $P_{d,m}(G)$.

PROOF: We can assume that $d \geq 2$, since the result is trivial for $d = 1$. Let G be a graph of order n that does not satisfy $P_{d,m}(G)$. Select vertices x and y for which there do not exists m vertex disjoint paths, each of length at most d, between the two vertices. Since G is m-connected, there are m vertex disjoint paths between x and y. Select m such paths with the sum of the lengths a minimum. Denote the paths by $P_1, P_2, \ldots, P_m$, and let $r_1 \leq r_2 \leq \ldots \leq r_m$ be the respective lengths of the paths. By assumption, $r_m > d$. Let H be the subgraph induced by the vertices on these m paths. Then $r = r_1 + \ldots + r_m - m + 2$ is the number of vertices in the subgraph H.

Assume $P_m = (x = v_0, v_1, \ldots, v_{r_m} = y)$. Since $r_m > d$, we can select the set $X = \{v_0, v_3, \ldots, v_{3(t-1)}\}$ with $t = \lfloor (d+4)/3 \rfloor$. We will determine an upper bound on the sum of the degrees of the vertices in X.

Any vertex not in H can be adjacent to at most a single vertex of X, for otherwise the path P_m could be shortened. So, the sum of the degrees of the vertices of x relative to $G - H$ is at most $n - r$.

The minimality of the length of the path P_m implies that there are no chords along the path. Therefore each vertex of X is adjacent to at most 2 vertices of P_m, and the sum of the degrees of the vertices in X relative to P_m is at most $2t \leq r_m + 3 - t$. In fact, unless x is adjacent to y, the right hand side of the inequality can be replaced by $r_m + 1 - t$.

Finally, if v_{3i} is adjacent to an internal vertex u of a path P_k for $k < m$, then v_{3j} for $j > i$ cannot be adjacent to any vertex of P_k that precedes u on the path (considered as starting at x and ending at y). Otherwise the paths P_m and P_k could be replaced by two paths of shorter length. Therefore, each interior vertex of P_k is adjacent to at most one vertex of X except for possibly $t - 1$ vertices (counting multiplicities). Thus, the sum of the degrees of X relative to the interior vertices of P_k is at most

$$r_k - 1 + (t - 1) = r_k + t - 2.$$

Note that if P_k contains just the edge xy, then there are no edges between X and the interior vertices of P_k, which yields a bound strictly less than $r_k + t - 2$. By taking this fact into account along with the effect the edge xy has on the number of edges between X and P_m, we have that the sum of the degrees of the vertices in X is bounded above by

$$(n-r) + (r_m+1-t) + \Sigma_{i=1}^{m-1} (r_i+t-2) + 1 \leq n - m + 2 + (m - 2)t.$$

Therefore, at least one of the t vertices in X has degree at most $\lfloor (n - m + 2)/t \rfloor + m - 2$. This gives a contradiction which completes the proof of Theorem 3. ∎

Before proving Theorem 4, we need to make some additional observations. It is not difficult to calculate $\mathrm{den}(H_1, s)$ for a specific $s \le (k - m + 1)\lfloor (d + 1)/2 \rfloor$. One critical observation needs to be made. Assume that S is a set of s vertices of H_1 which induces a minimum number of edges of H_1. We can certainly assume that S is a subset of the vertices in the rim of the generalized wheel.

Let A, B, C and D be the vertices of S in four consecutive sets of vertices along the rim of H_1, and let a, b, c and d be the orders of the sets respectively. The number of edges in the graph induced by these vertices is

$$N_1 = \binom{a}{2} + \binom{b}{2} + \binom{c}{2} + \binom{d}{2} + ab + bc + cd.$$

Considering the case when $a \ge d$ and $b \ge 1$, and moving one vertex from B to C, a set S' is obtained with s vertices that induces a subgraph with the following number of edges:

$$N_2 = \binom{a}{2} + \binom{b-1}{2} + \binom{c+1}{2} + \binom{d}{2} + a(b-1) + (b-1)(c+1) + (c+1)d.$$

Since $N_1 - N_2 = a - d$, the minimum is also attained with S'. In fact, there is no loss of generality in assuming that $b = 0$. Repeated application of this observation and elementary properties of binomial coefficients gives that the minimum will occur for a set S in which the number of vertices in alternating sets on the rim is 0, and the number in the remaining sets differ by at most 1 (with the exception of the two sets of rim vertices contained in the complete graphs with a single vertex).

The previous observations reduce the calculation of $den(\mathcal{H}_1,s)$ to the determination of the number of edges in $\lfloor(d + 1)/2\rfloor$ copies of complete graphs each with either $\lfloor s/\lfloor(d + 1)/2\rfloor\rfloor$ or $\lceil s/\lfloor(d + 1)/2\rfloor\rceil$ vertices. It should be noted that some minor adjustments must be made in this calculation to account for the fact that a minimum set S can contain vertices in the two singleton vertex sets on the rim. However, the order of magnitude of $den(\mathcal{H}_1,s)$ is $s\lfloor s/\lfloor(d+1)/2\rfloor\rfloor)/2$. Therefore, the density of $\mathcal{H}_1$ implies that the density condition in Theorem 4 has the correct order of magnitude, and the constant factor cannot be reduced by more than a factor of 2/3.

We need the following technical lemma to prove Theorem 4. The essential observation of the proof of Lemma 5 was used in the previous proof. However, since the fact will be applied repeatedly in the proof of Theorem 4, it is stated separately for easy reference.

LEMMA 5: Let P and Q be internally vertex disjoint paths from x to y in a graph G, such that their sum of their lengths is a minimum. If A and B are subsets of vertices of P and Q respectively, such that A does not contain any pair of consecutive vertices on P, then the number of edges betweeen A and B is at most $|A| + |B| - 1$.

PROOF: Let $A = \{a_1, a_2, \ldots, a_p\}$ and $B = \{b_1, b_2, \ldots, b_q\}$ with the vertices listed in the order they appear on the path. If a_r is adjacent to b_s, then a_i is not adjacent to b_j for any $i > r$ and $j < s$; otherwise, there would be a pair of paths from x to y with the sum of their lengths less than that of P and Q. Therefore, each vertex of B will be adjacent to at most 1 vertex of A, except for possibly

$|A| - 1$ (counting multiplicities) vertices. This proves Lemma 5. ∎

THEOREM 4: Let m and $d \geq 2$ be positive integers, and let G be a k-connected graph $(k \geq m)$ of order n. If there exists some s $(s \leq (k-m+1)\lfloor d/2 \rfloor)$ such that G satisfies the inequality

$$den(G,s) \geq (3/4)s \lceil s/\lfloor d/2 \rfloor \rceil,$$

then G has property $P_{d,m}(G)$.

PROOF: Assume G does not satisfy $P_{d,m}(G)$, and let x and y be a pair of vertices for which the appropriate paths do not exist. Select k internally vertex disjoint paths, say $P_1, P_2, \ldots, P_k$, from x to y so that the sum of the lengths of the paths is a minimum. Assume that the lengths of the paths are $p_1 \leq p_2 \leq \ldots \leq p_k$ respectively. By assumption we have $p_i > d$ for all $i \geq m$.

For each $i \geq m$, let A_i be the first $2\lfloor d/2 \rfloor$ internal vertices on P_i, and A_i^o and A_i^e the $\lfloor d/2 \rfloor$ odd and even indexed vertices of A_i respectively. Let $r = \lceil s/\lfloor d/2 \rfloor \rceil$. We will select a set S of s vertices in G, by selecting either A_i^o or A_i^e for each i $(m \leq i \leq m+r-1)$. Actually, only a proper subset of A_{m-r+1}^o or A_{m-r+1}^e will be selected unless s is divisible by $\lfloor d/2 \rfloor$. An upper bound on the number of edges in the subgraph spanned by S will be determined.

The selection of S will be done by induction on i

$(m \le i \le m+r-1)$, and in such a way as to minimize the number of edges in the induced subgraph. From Lemma 5 there are less than $3 \lfloor d/2 \rfloor$ edges between A_m^o and A_{m+1}, so the number of edges between A_m^o and either A_{m+1}^o or A_{m+1}^e, say A_{m+1}^o, is less than $3 \lfloor d/2 \rfloor/2$. Assume the appropriate subset of A_i, which we can say with no loss of generality is A_i^o, has been selected for all $i < j$. Repeated application of Lemma 5 implies that the number of edges between A_j and the union of the A_i^o for $i < j$ is less than $3(j - m) \lfloor d/2 \rfloor$. Hence, for an appropriate selection of a subset of A_j, say A_j^o, the number of edges between this subset and the previously chosen subsets is at most $(3/2)(j - m) \lfloor d/2 \rfloor$. Therefore, by induction, S can be chosen such that the number of edges in the induced subgraph is less than

$$(3/2) \lfloor d/2 \rfloor (1 + 2 + \dots + (r-1)) < (3/4)s \lceil s/ \lfloor d/2 \rfloor \rceil.$$

This contradiction completes the proof of Theorem 4. ∎

Note that for $s \le \lfloor d/2 \rfloor$, the inequality in the statement of Theorem 4 is $\text{den}(G,s) > 0$. Theorem 4 for this special case comes from the trivial observation that any set of vertices that are not consecutive on a minimal length path cannot be adjacent. If the path has length greater than d, there exist $\lfloor d/2 \rfloor$ such vertices.

4. PROBLEMS

There are several natural and interesting questions related to the results presented that have not been answered. The density function in Theorem 4 has the same order of magnitude as the density function den(n,k,s) defined from the generalized wheel graphs $\mathcal{H}_1$ and $\mathcal{H}_2$. Can one show for a fixed s that any k-connected graph G of order n such that each induced subgraph on s vertices has more than den(n,k,s) edges satisfies $P_{d,m}(G)$?

Minimal connectivity and degree conditions that separately imply $P_{d,m}$ have been investigated. Also considered was the minimal degree condition sufficient for $P_{d,m}$ assuming the graph was m-connected. However, it is natural to expect that with higher connectivity, the degree condition needed to imply $P_{d,m}$ would be less. This relationship between these two parameters and what is needed to give $P_{d,m}$ should be studied.

Recently, classical results on paths and cycles in graphs that were based on degree conditions, such as minimal degree and the sum of the degrees of nonadjacent vertices, have had neighborhood condition analogs (see [4] and [5]). One such neighborhood condition involves the minimum number of vertices in the union of the neighborhoods of each pair of nonadjacent vertices of the graph. Recall, that in the generalized wheel graph W(m-2,n-m+2;1) the order of the union of a nonadjacent pair of vertices can be as small as $m + 1$, while in the m-cube it is at least $2m - 2$. However, the minimum degree and the minimum sum of the degrees of nonadjacent vertices is the same for the two graphs. This indicates that the relationship between neighborhood conditions and property $P_{d,m}$ should be investigated, because neughborhood conditions might yield sharper results than degree conditions.

Also, one need not consider only nonadjacent pairs of vertices, but adjacent pairs or all pairs of vertices can be considered in neighborhood conditions implying $P_{d,m}$. In addition, one need not restrict consideration to just the union of the neighborhoods of pairs of vertices. For any fixed integer t, the number of vertices in the union of the neighborhoods of any set of t (nonadjacent) vertices can be considered in the neighborhood condition. For all of these possibilities of a neighborhood condition, the generalized wheel and the cube have significantly different properties, so many interesting problems remain.

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