No Three Faces with the Same Number of Edges: 10856

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11206. *Proposed by Mircea Ivan, Technical University of Cluj-Napoca, Cluj-Napoca, Romania, and Alexandru Lupaş, University of Sibiu, Sibiu, Romania.* Find

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left\{ \frac{n}{k} \right\}^2 ,$$

where $\{x\}$ denotes $x - \lfloor x \rfloor$, the fractional part of x.

11207. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.* Let $\langle a_n \rangle$ be a sequence of distinct real numbers with the property that for each $\epsilon > 0$ there exists $\eta > 0$ such that for all positive integers n and m

$$\epsilon \le |a_n - a_m| < \epsilon + \eta \qquad \Longrightarrow \qquad |a_{n+1} - a_{m+1}| < \epsilon.$$

Prove that $\langle a_n \rangle$ converges to a (finite) limit.

SOLUTIONS

No Three Faces with the Same Number of Edges

10856 [2001, 172]. *Proposed by Andrei Jorza, "Moise Nicoara" High School, Arad, Romania.*Find all bounded convex polyhedra such that no three faces have the same number of edges.

Composite solution by Edward Ordman, University of Memphis, Memphis, TN; Richard Stong, Rice University, Houston, TX; and the editors. **Necessary conditions** for such polyhedra. Begin with Euler's formula $V - E + F = 2$. Let F_i denote the number of faces with i edges. Note that $F_1 = F_2 = 0$, so

$$F = F_3 + F_4 + F_5 + F_6 + \cdots .$$

Since each edge is on exactly two faces,

$$2E = 3F_3 + 4F_4 + 5F_5 + 6F_6 + \cdots .$$

Each vertex is on at least three faces, so

$$3V \le 3F_3 + 4F_4 + 5F_5 + 6F_6 + \cdots ,$$

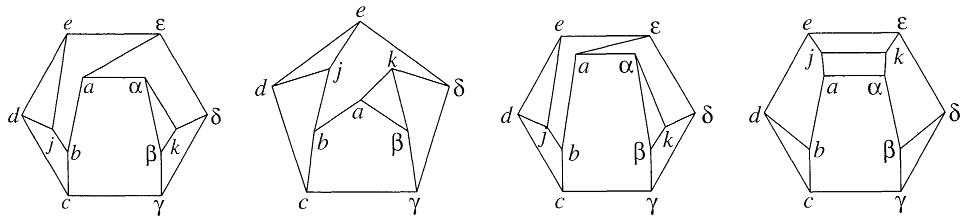
with equality only if each vertex is on exactly three faces. Thus

$$\begin{aligned} 12 = 6V - 6E + 6F &\le 2\big(3F_3 + 4F_4 + 5F_5 + \cdots\big) \\ &\quad - 3\big(3F_3 + 4F_4 + 5F_5 + \cdots\big) + 6\big(F_3 + F_4 + F_5 + \cdots\big) \\ &= 3F_3 + 2F_2 + F_5 + 0F_6 - F_7 - 2F_8 - \cdots . \end{aligned}$$

Since $F_i \le 2$ for all i, we can achieve the total 12 only if $F_3 = F_4 = F_5 = 2$ and $F_i = 0$ when $i \ge 7$. The only term still not known is F_6, which can be 0, 1, or 2. In all three of these cases, we have equality in the inequalities, so each vertex is on exactly three faces, and therefore on exactly three edges. Therefore two faces that are not disjoint must have exactly one edge in common. Also, if three edges meet in a vertex, then any face that contains that vertex must contain exactly two of those edges.

Each bounded and convex polyhedron produces a planar graph by stereographic projection; the graph has all the combinatorial properties of the original polyhedron.

We construct these graphs by starting from two contiguous faces and adjoining, sequentially, a new face to an existing one. A face with three, four, five, or six edges will be called a **3**, a **4**, a **5**, or a **6**, respectively.



Graph 1 Graph 2 Graph 3 Graph 4

Case 1: $F_6 = 0$. There are eight vertices. The two **5**s cannot be disjoint, because that would require ten vertices, so they have one edge in common. Let one of the **5**s be the unbounded face. Label it $abfgh$, and let the other **5** be $abcde$. These are the only vertices. The other faces are in the region $bcdeahgfb$. Any face that has b as a vertex must contain c and f (but none of the other vertices) and is therefore bcf, a **3**. By the same argument, aeh is the other **3**. One of the **4**s has cf as an edge, must contain d, g but none of a, e, or h, and is therefore $cdgf$. The four remaining vertices yield the remaining **4**.

Thus the case $F_6 = 0$ yields **Graph 1**.

Case 2: $F_6 = 1$. There are ten vertices. The two **5**s cannot be disjoint, since then the **6** would have at least three vertices in common with one of them. The two **5**s thus have one edge in common, so together they account for eight of the ten vertices. Label the two additional vertices j and k. Let one of the two **5**s be the unbounded face, label it $cde\delta\gamma$, and label the other **5** $abc\gamma\beta$.

Let F be the third face with vertex c. It must contain b and d but no other vertex of either **5**. Since F also has at most j and k outside the **5**s, F cannot be the **6**, so F is a **4** or a **3**. If F is a **4**, then F must be $bcdj$ or $bcdk$. Exchanging the labels j and k if necessary, we may assume F is $bcdj$. If F is a **3**, then F is bcd. The other face G that contains bd must also contain a and e, (but not β, γ, or δ) and must be a **6** or a **4**. If G is a **6**, then G is $abdejk$ or $abdekj$. If G is a **4**, then G is $abde$.

To summarize: starting from vertex c we get three possibilities: (i) $bcdj$, (ii) bcd, $abdejk$, and (iii) bcd, $abde$. Similarly, starting from vertex γ we are no longer free to switch labels j and k, so we will get these possibilities: (i_1) $\beta\gamma\delta j$, (i_2) $\beta\gamma\delta k$, (ii_1) $\beta\gamma\delta$, $a\beta\delta ejk$, (ii_2) $\beta\gamma\delta$, $a\beta\delta ekj$, and (iii_1) $\beta\gamma\delta$, $a\beta\delta e$. Next we must pair each of the three possibilities from the first list with each of the five possibilities from the second list.

(i)(i_1) would give four edges on j, so it is impossible.

For the pairing (i)(i_2) the face F that contains d must contain e and j but not b and δ; it must be a **6** or a **3**. If F were the **6**, then it would contain the third edge from e, for which the only choices would be ea, ej, or ek; ea or ej would make F a **5** or a **3**, and ek would leave a with only two edges. If F is a **3**, then F is edj; the other face G with edge ej contains b, a, δ, k, so G is the **6** labeled $ejbak\delta$ and the other **3** is $ak\beta$. This is **Graph 2**.

(i)(ii_1) and (i)(ii_2) would give more than three edges on j, so they are impossible.

(i)(iii_1) The region $edjbae$ would surround k and leave no possibility for the third edge on j, so this is impossible.

(ii)(i_1) and (ii)(i_2) are relabeled versions of (i)(ii_1) and (i)(ii_2).

(ii)(ii_1) and (ii)(ii_2) have two coplanar **6**s, so they are impossible.

(ii)(iii_1) would give fewer than 3 edges to j and k, making it impossible.

(iii)(i$_1$), (iii)(ii$_1$), and (iii)(ii$_2$) are relabeled versions of previous cases.

(iii)(iii$_1$) has four edges from e, so it is impossible.

Thus the case $F_6 = 1$ yields **Graph 2**.

Case 3: $F_6 = 2$. There are twelve vertices. If the two **6**s were disjoint, then they would account for all of the vertices, and each **5** would have at least three vertices in common with one of the **6**s. So the two **6**s have one edge in common. Let one of the **6**s be the unbounded face, label it $cde\varepsilon\delta\gamma$, label the other **6** as $abc\gamma\beta\alpha$, and let j and k be the two remaining vertices. Each **5** has at most two vertices at j and k, so at least three vertices on the two **6**s together. Accordingly, each **5** has an edge in common with each **6**. Let F be the third face with vertex c. It may be a **5**, a **4**, or a **3**; it must contain b and d, but none of the other vertices on the two **6**s.

If F were a **5**, then it would necessarily be $bcdjk$ (or $bcdkj$, but we may switch the labels j and k). Let G be the other face with vertex d; it must contain e and j, but any additional vertex would be on four edges, showing that G is edj. Similarly, the third face with vertex b must be abk. The other face at e would necessarily contain $\varepsilon ejka\alpha$ and be a **6**. So this case (F is a **5**) is impossible.

If F is a **4**, then it is $bcdj$ (or $bcdk$, but we may switch the labels j and k). Let G be the third face at b, which must contain a and j but neither d nor α. If G is a **5**, then it has an edge in common with the unbounded **6**. If that edge were $\varepsilon\delta$ then e would be on only two edges. It follows that the edge is $e\varepsilon$ and G is **5** labeled $ejba\varepsilon$, which forces the face dej. Call this possibility (i). If G is a **4**, then it is $ejba$ (which would give too many edges to e) or $kjba$ (which would imply $kjde$ and give three **4**s). If G is a **3**, then it is abj, which case is seen to be symmetric by exchanging the two **6**s to case (i) treated earlier.

If F is a **3**, then it is bcd. Any face G that contains bd must contain a and e, but none of the other vertices of the **6**s. If G is a **5**, then it could be $abde\varepsilon$ or $abde\alpha$, both of which would leave a vertex with only two edges, or $abdej$ (or $abdek$). Call this possibility (ii). If G is a **4**, then it is $abde$. Call this possibility (iii).

Therefore, starting with vertex c we have three possibilities: (i) $bcdj$, $ejba\varepsilon$, dej, (ii) bcd,$abdej$, and (iii) bcd, $abde$. Similarly, starting from γ we are no longer free to switch the labels j and k or to switch the two **6**s, so we get these possibilities: (i$_1$) $\beta\gamma\delta j$, $\varepsilon j\beta\alpha e$, $\delta\varepsilon j$, (i$_2$) $\beta\gamma\delta k$, $\varepsilon k\beta\alpha e$, $\delta\varepsilon k$, (i$_3$) $\delta\gamma\beta j$, $\alpha j\delta\varepsilon a$, $\beta\alpha j$, (i$_4$) $\delta\gamma\beta k$, $\alpha k\delta\varepsilon a$, $\beta\alpha k$, (ii$_1$) $\beta\gamma\delta$, $\alpha\beta\delta\varepsilon j$, (ii$_2$) $\beta\gamma\delta$, $\alpha\beta\delta\varepsilon k$, and (iii$_1$) $\beta\gamma\delta$, $\alpha\beta\delta\varepsilon$. Next we must pair each of the three possibilities in the first list with each of the seven possibilities in the second list.

The pairings (i)(i$_1$), (i)(i$_3$), (i)(ii$_1$), and (ii)(ii$_1$) leave j with too many edges, so they are impossible.

The pairing (i)(i$_2$) crosses αe and $a\varepsilon$, so it is impossible.

(i)(i$_4$) yields **Graph 3**.

(i)(ii$_2$) and (i)(iii$_1$) leave ε with too many edges, so they are impossible.

(ii)(ii$_2$) leaves j and k with two edges, which must be joined with an edge and yield **Graph 4**.

(ii)(iii$_1$) leaves k with no edges and j with only two edges, so it is impossible.

(iii)(iii$_1$) leaves k and j with no edges, so it is impossible.

Thus the case $F_6 = 2$ yields **Graph 3** and **Graph 4**.

Sufficient conditions. We must also show that each of the four graphs pictured actually corresponds to a geometric polyhedron. For Graph 1, truncate a tetrahedron at two vertices. For Graph 2, begin with a triangular prism and truncate it at two diagonally opposed vertices of a quadrangular face. For Graph 3, begin with Graph 2 and truncate it at a vertex where a triangle, quadrilateral, and pentagon meet. For Graph 4, truncate a cube at two vertices with a common edge.

Editorial comment. Some solvers presented the following generalization: *If d is an integer greater than 1, then there are a finite number of polyhedra (up to placement of vertices) such that no $d + 1$ faces have the same number of edges.* References provided: A. J. W. Duijvestijn & P. J. Federico, "The Number of Polyhedral Graphs," *Math. Comput.* **37** (1981), 523-532; P. Engel, "On the Enumeration of Polyhedra," *Discrete Math.* **41** (1982), 215-218; G. M. Zeigler, *Lectures on Polytopes* (Springer-Verlag, 1998).

Also solved by S. Amghibech (Canada), R. Chapman (U. K.), D. Donini (Italy), C. C. Heckman, L. Zhou, and the GCHQ Problem Solving Group (U. K.).

The Limit of a Set-Valued Process

11052 [2003, 957]. *Proposed by Danrun Huang and Daniel Scully, St. Cloud State University, Saint Cloud, MN.* Let $\mathcal{P}_n$ be the set of all subsets of $\{1, \dots, n\}$, and let $\Phi\colon \mathcal{P}_n \to \mathcal{P}_n$ be given by

$$\Phi(S) = \begin{cases} \{1\} \cup S & \text{if } 1 \notin S, \\ \{1, \dots, n\} \backslash \{k - 1\colon\ k \in S \text{ and } k > 1\} & \text{if } 1 \in S. \end{cases}$$

Let $M(S)$ denote $\sum_{k \in S} k$. Given that n is a positive integer and that $S \in \mathcal{P}_n$, prove that the following limit exists, and evaluate it:

$$\lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^{m} M(\Phi^j(S)).$$

Solution by Achava Nakhash, Torrance, CA. The limit is $(n + 1)^2/4$ when n is odd and $(n^2 + 2n + 2)/4$ when n is even.

Given n, let $S^* = \{n - 2i\colon\ 0 \le i < \lfloor n/2 \rfloor\}$. We show first that for each S in $\mathcal{P}_n$ the process reaches S^*. Call each iteration when 1 is added a *unit step*, and call the other iterations *sliding steps*. Each unit step is followed by a sliding step, after which n is present. Unit steps do not affect the presence of any number other than 1, so n remains after it first appears. The next sliding step eliminates $n - 1$ (if present), after which it cannot reappear. After the next sliding step, $n - 2$ is present, and then $n - 3$ is eliminated. By repeating this argument we arrive at S^*.

Since $\Phi(S^*) = S^*$ when n is odd, in this case the limit is $M(S^*)$, which equals $(n + 1)^2/4$. When n is even, $M(S^*) = S^* \cup \{1\}$ and $M^2(S^*) = S^*$, so the limit is $\frac{1}{2}[M(S^*) + M(S^* \cup \{1\})]$, which equals $(n^2 + 2n + 2)/4$.

Editorial comment. The published problem had "S is a nonempty subset of $\mathcal{P}$" instead of $S \in \mathcal{P}_n$. Here "subset" should be "element," the subscript "n" should be present, and the exclusion of $\emptyset$ is unnecessary, since $\Phi(\emptyset) = \{1\}$. Uroś Milutinović expressed the process numerically, viewing S as the binary expansion of an integer, and generalized the result in that setting.

Also solved by R. A. Agnew, S. Amghibech (Canada), R. B. Bagley, D. Beckwith, P. Budney, J. Caffrey & R. Jayne, P. P. Dályay, R. DiSario, M. Dolatabady (Iran), D. Donini (Italy), N. Dukich, J.-P. Grivaux (France), E. A. Herman, J. H. Lindsey II, O. P. Lossers (Netherlands), U. Milutinović (Slovenia), D. K. Nester, G. Raduns, M. Reyes, M. Spivey, N. C. Singer, R. Stong, L. Zhou, BSI Problems Group (Germany), the GCHQ Problem Solving Group (U. K.), NSA Problems Group, University of Louisiana–Lafayette Math Club, and the proposers.