Catalan Numbers

E 2383 [1972, 1034]. *Proposed by E. T. Ordman, University of Kentucky*
Let n be a nonnegative integer. For $p = 1, 2, \cdots$ define

$$S_p(n) = \sum_{k=0}^{[n/2]} \left[\binom{n}{k} - \binom{n}{k-1}\right]^p,$$

where we make the usual conventions regarding binomial coefficients. It is easy to evaluate $S_1(n)$. Evaluate $S_2(n)$.

Solution by Richard Gibbs and Harold Stocker, Fort Lewis College, Durango, Colorado. Let

$$T(n) = \sum_{k=0}^{n+1} \left[\binom{n}{k} - \binom{n}{k-1}\right]^2.$$

By the symmetry of the binomial coefficients, $T(n) = 2S_2(n)$. Now

$$T(n) = \sum_{k=0}^{n+1} \left[\binom{n}{k}^2 - 2\binom{n}{k}\binom{n}{k-1} + \binom{n}{k-1}^2\right].$$

From the identity $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k}$ we obtain

$$2\binom{n}{k}\binom{n}{k-1} = \binom{n+1}{k}^2 - \binom{n}{k}^2 - \binom{n}{k-1}^2.$$

Therefore,

$$T(n) = 2\sum_{k=0}^{n+1} \binom{n}{k}^2 + 2\sum_{k=0}^{n+1} \binom{n}{k-1}^2 - \sum_{k=0}^{n+1} \binom{n+1}{k}^2.$$

Using the identity $\sum_{j=0}^{m} \binom{m}{j}^2 = \binom{2m}{m}$ and the fact that $\binom{n}{-1} = \binom{n}{n+1} = 0$ we obtain

$$T(n) = 4\binom{2n}{n} - \binom{2n+2}{n+1} = 4\binom{2n}{n} - 2\left[\binom{2n}{n-1} + \binom{2n}{n}\right]$$

$$= 2\binom{2n}{n} - 2\binom{2n}{n-1}.$$

Therefore

$$S_2(n) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

Thus $S_2(n)$ is just the nth Catalan number.

Also solved by Günter Bach (Germany), Anders Bager (Denmark), M. T. Bird, D. M. Bloom, Dieter Bode (Germany), Frederick Carty, H. W. Gould, M. G. Greening (Australia), Robert Heller, J. D. Hiscocks, O. P. Lossers (Netherlands), Alexandru Lupas (Romania), Milan Lustig (Czechoslovakia), Kumer Murty & Ram Murty, M. R. Railkar (India), Jürg Rätz (Switzerland), Kenneth Rosen, F. C. Smith, Phil Tracy, E. Trost (Switzerland), David Zeitlin, and the proposer.