Distinct Representatives for a Collection of Finite Sets

E 2309 [1971, 792]. *Proposed by Václav Chvátal, University of Waterloo, Ontario*

Prove the following: Let $A_1, \cdots, A_n$ be finite sets. If

$$\sum_{1 \leqq i < j \leqq n} |A_i \cap A_j| \frac{|A_1| \cdots |A_n|}{|A_i| \cdot |A_j|} < 1,$$

then the sets $A_1, \cdots, A_n$ have a system of distinct representatives (i.e., there are $a_1, a_2, \cdots, a_n$ such that $a_i \in A_i$ and $a_i \neq a_j$ for $i \neq j$).

Solution by the proposer. There are exactly $|A_1| \cdot |A_2| \cdots |A_n|$ mappings

$$f: \{1, 2, \cdots, n\} \to \bigcup_{i=1}^{n} A_i \tag{1}$$

such that $f(i) \in A_i$ for $i = 1, 2, \cdots, n$. The problem asks if there exists a function of type (1) which is one-to-one. If a function h is of type (1) and is not one-to-one, then there are at least two distinct integers $i, j \in \{1, 2, \cdots, n\}$ such that $h(i) = h(j)$. The number of mappings of type (1) which are not one-to-one then does not exceed

$$\sum_{1 \leqq i < j \leqq n} |A_i \cap A_j| \frac{|A_1| \cdot |A_2| \cdots |A_n|}{|A_i| \cdot |A_j|}$$

$$= |A_1| \cdot |A_2| \cdots |A_n| \sum_{1 \leqq i < j \leqq n} \frac{|A_i \cap A_j|}{|A_i| \cdot |A_j|}$$

$$< |A_1| \cdot |A_2| \cdots |A_n| \quad \text{(by hypothesis).}$$

Therefore, the number of functions of type (1) which are not one-to-one is strictly less than the number of functions of type (1) and hence the conclusion follows.

Also solved by the Bennett College Team, D. M. Bloom, Bobby Chapuis & C. C. Rousseau, John Christopher, M. G. Greening (Australia), David Kelly, Harry Lass, Robert Patenaude, David Sumner, and J. H. Timmermans (Netherlands).

A Categorical Impossibility

E 2310 [1971, 793]. *Proposed by Hal Forsey, San Francisco State College*

Does there exist a positive function f such that if x is rational and y is irrational, then $f(x)f(y) \leqq |x - y|$?

I. *Solution by Simeon Reich, Israel Institute of Technology, Haifa.* The answer is no. Let R, I, and Q denote the reals, irrationals, and rationals respectively, and suppose that the desired f exists. We note first that if $\{r_n\}$ is a sequence of rationals which converges to an irrational, then $f(r_n) \to 0$, and likewise if $\{y_n\}$ is a sequence of irrationals which converges to a rational, then $f(y_n) \to 0$. Now let $g: R \to R$ agree with f on I and vanish on Q. Then Q is the set of points where g is continuous, so that it must be a G_δ set. But it is not, by the Baire Category Theorem.

Alternatively, we can let $h: R \to R$ agree with f on Q and vanish on I. Since h is Lipschitzian on I and discontinuous on Q, I must be a set of the first category by Theorem 4 of G. A. Heuer, *A property of functions discontinuous on a dense set*, this MONTHLY, 73 (1966), 378–379. But it is not, again by the Baire Category Theorem.

II. *Solution by Pavel Kostyrko, Bratislava, Czechoslovakia.* We characterize such functions in the following theorem.

THEOREM. *Let (M,d) be a metric space, and suppose that $X \subseteq M$ and $Y = M \setminus X$. Then there exists a (strictly) positive function $f: M \to R$ such that*

$$f(x)f(y) \leqq d(x,y) \text{ for all } x \in X,\ y \in Y \tag{1}$$

if and only if both X and Y are F_σ sets in M. [Note that if $X = \varnothing$ or $Y = \varnothing$, then (1) is vacuously satisfied—Ed.]

Proof. Suppose that such a function f exists and let $X_n = \{x \in X: f(x) \geqq 1/n\}$ for $n = 1, 2, \cdots$. We show that $\bar{X}_n \subseteq X$ for all n, where $\bar{Z}$ denotes the closure of Z in M. Suppose to the contrary that there exists a positive integer m and a y such that $y \in \bar{X}_m \setminus X$. Then $y \in Y$ and there exists a sequence $\{x_k\}$ of elements of X_m such that $x_k \to y$. Whence $f(y)/m \leqq f(x_k)f(y) \leqq d(x_k, y) \to 0$, implying that $f(y) = 0$, a contradiction. It follows that

$$X = \bigcup_{n=1}^{\infty} X_n \subseteq \bigcup_{n=1}^{\infty} \bar{X}_n \subseteq X,$$

so that X is an F_σ set in M. The proof for Y is analogous.

Conversely, suppose that X and Y are F_σ sets in M. Write $X = \bigcup_{n=1}^{\infty} F_n$ and $Y = \bigcup_{n=1}^{\infty} F_n^*$, where F_n and F_n^* are closed for $n = 1, 2, \cdots$, and where we assume without loss of generality that $F_1 \subseteq F_2 \subseteq \cdots$, and $F_1^* \subseteq F_2^* \subseteq \cdots$. The function f is defined as follows: If $x \in X$, let $n(x)$ denote the least positive n such that $x \in F_n$. Then define $f(x) = \min\{d(x, F_{n(x)}^*), 1\}$. If $y \in Y$, define $f(y)$ analogously. It can then be verified by checking cases that f has the required properties.

The problem is now solved by noting that the set of irrationals in R with the usual metric is not an F_σ set by the Baire Category Theorem.

III. *Solution by Charles Schelin, Wisconsin State University, La Crosse.* The answer is no. Suppose, to the contrary, that such a function exists. Let Q denote the set of rationals and H the set of irrationals. We note that if x is irrational and y is rational (or vice versa) then

$$f(x) \leqq \frac{1}{f(y)} |x - y|. \tag{*}$$

Let I_0 be any compact interval. Choose $x_1 \in H \cap I_0^o$, where I_0^o is the interior of I_0. By (*) we can find a neighborhood $N_1 = (x_1 - \delta_1, x_1 + \delta_1)$ of x_1 such that if $y \in Q \cap N_1$, then $f(y) < 1$. Now choose a fixed $y_1 \in Q \cap N_1 \cap I_0^o$; again by (*) we can find a neighborhood $M_1 = (y_1 - \eta_1, y_1 + \eta_1)$ of y_1 such that if $x \in H \cap M_1$, then $f(x) < 1$. Then for all $t \in M_1 \cap N_1 \cap I_0^o$ it is true that $f(t) < 1$. Select a nontrivial closed interval $I_1 \subseteq M_1 \cap N_1 \cap I_0^o \subseteq I_0$.

Continuing this process, we obtain a nested sequence $I_0 \supseteq I_1 \supseteq \cdots$ of closed bounded intervals with $f(t) < 1/n$ for all $t \in I_n$. By the Nested Interval Theorem, there is some $w \in \bigcap_{n=1}^{\infty} I_n$, forcing $f(w) < 1/n$ for every n; hence $f(w) \leqq 0$, a contradiction.

Also solved by Sheldon Axler, Bill Beckmann, Harold Donnelly, Neal Felsinger, Peter Frankl (Hungary), Gary Gunderson, G. A. Heuer, Terjéki József (Hungary), Peter Kuhfittig, Harry Lass, P. L. Montgomery, E. T. Ordman, Wolfe Snow, David Sumner, and the proposer.

Editorial Comment. Schelin's construction can be generalized as follows: Suppose that M is a compact metric space, and that Q is a subset of M such that both Q and its complement H are dense in every ball of positive radius. Then there cannot exist a strictly positive real-valued function f on M such that $f(x)f(y) \leqq d(x, y)$ for every $x \in Q$ and $y \in H$. Schelin's proof is interesting because it does not explicitly use the Baire Category Theorem.

The Compleat Cyclic Quadrilateral

E 2311 [1971, 793]. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Prove that, if a quadrilateral $A_1A_2A_3A_4$ can be inscribed in a circle, then the (six) lines drawn from the midpoints of A_pA_q perpendicular to A_rA_s (p, q, r, s distinct) are concurrent.

Solution by Sister Stephanie Sloyan, Georgian Court College, Lakewood, N.J. Assume that the circle is the unit circle and identify the point A_i with the complex number a_i in the usual manner. Then the line from the midpoint of the segment A_pA_q perpendicular to A_rA_s is given by

$$z - a_r a_s \bar{z} = \tfrac{1}{2}(a_p + a_q) \, \frac{a_p a_q - a_r a_s}{a_p a_q},$$

and it is easily calculated that all six lines pass through the point $\frac{1}{2}(a_1 + a_2 + a_3 + a_4)$. J. W. Clawson, *The complete quadrilateral*, Annals of Math. 20 (1918–1919), 232–261, calls this point the *orthic center* of the quadrilateral.

In a similar fashion one can show that the three lines joining the midpoint of A_pA_q to that of A_rA_s (p, q, r, s distinct) are each bisected by a point identified by Clawson as the *mean center* of the quadrilateral. Since the mean center is given by $\frac{1}{4}(a_1 + a_2 + a_3 + a_4)$, it follows that it lies halfway between the orthic center and the circumcenter.