$$\lim \sum_{1}^{\infty} \left(\frac{x}{n} \right)^{n}$$

E 2928 [1982, 130]. *Proposed by Doug Wiedemann, Institute for Defense Analysis.*

Find $\lim(\Sigma_1^\infty x^n/n^n)$, $x \to -\infty$.

Solution by Mark Pinsky, Northwestern University, Evanston, IL. For any x,

$$\sum_{n=1}^{\infty} x^n/n^n = x\int_0^1 e^{xu\ln(1/u)}\,du.$$

(This can be seen by expanding the exponential in a power series, uniformly convergent for $0 \leqslant u \leqslant 1$, and integrating term by term.)

If $x < 0$, the integrand is less than 1, except at $u = 0$ and $u = 1$. For any $\delta > 0$, when $x \to -\infty$ integrating by parts yields

$$\int_0^{\delta} e^{xu\ln(1/u)}\,du = o(1/x)$$

$$\int_{\delta}^{1-\delta} e^{xu\ln(1/u)}\,du = O(1/x^2),$$

and

$$\int_{1-\delta}^{1} e^{xu\ln(1/u)}\,du = -1/x + O(1/x^2).$$

Therefore $\Sigma_{n=1}^{\infty} x^n/n^n \to -1$ when $x \to -\infty$.

Also solved by P. F. Byrd, M. L. Glasser, J. E. Jamison and C. C. Rousseau, O. P. Lossers (The Netherlands), L. E. Mattics, W. A. Newcomb, O. G. Ruehr, M. El-Tohami, and the proposer.

Equations Forcing Commutativity

E 2931 [1982, 131]. *Proposed by Chen-Te Yen, Chung Yuan Christian University, Chung-Li, Taiwan.*

Let n, m be fixed positive integers. Suppose G is a group such that for all x, y in G, $(xy)^n = x^n y^n$, $(xy)^{n+m} = x^{n+m} y^{n+m}$, and $(xy)^{n+2m} = x^{n+2m} y^{n+2m}$. Show that G is abelian if $m = 1$ or 2. What about $m \geqslant 3$?

Solution by I. N. Herstein, University of Chicago, and Rony Teitler, Wolfson College, Oxford, England. The solvers call attention to F. W. Levi, *Notes on group theory*, J. Indian Math. Soc. 8, (1944) 1–9.

THEOREM (Levi). *If* $(xy)^n = x^n y^n$ *for all* x, y *in a group* G *and all* n *in a set* S *of integers, then* G *is forced to be commutative if and only if* $\gcd\{n^2 - n,\ n \in S\}$ *is* 2. (See also this MONTHLY, E2411; 1974, 410.)

Teitler referred also to Ronse, Teitler, *Groupes dans lesquels l'élévation à une puissance entière est un endomorphisme*, Bull. Acad. Roy. Belg. 5e sér., 62 (1976) 539–564.

Teitler proved also the following results, which he believes to be new.

1. If G is a group in which the maps $x \to \phi(x)$, $x \to x\phi(x)$, $x \to x^2\phi(x)$ are endomorphisms of G, then G is abelian.

2. If $x \to \phi(x)$, $x \to x^2\phi(x)$, $x \to x^4\phi(x)$ are endomorphisms, then G is abelian.

Proof of 1. Set $B(x) = x\phi(x)$. From $B(xy) = B(x)B(y)$ it follows that $xy\phi(x)\phi(y) =$

$x\phi(x)y\phi(y)$, so that $y\phi(x) = \phi(x)y$, and $\phi(x)$ must always lie in the center of G.
Define $\Gamma(x) = x^2\phi(x) = xB(x)$. Then $x\phi(x)$ is in the center. Thus x is in the center.

Proof of 2. Set $B(x) = x^2\phi(x)$. Then $B(xy) = B(x)B(y)$ gives $(xy)^2\phi(x)\phi(y) = x^2\phi(x)y^2\phi(y)$, so that

$$(*) \qquad yxy\phi(x) = x\phi(x)y^2.$$

Set $\Gamma(x) = x^4\phi(x) = x^2B(x)$. Then $yxyB(x) = xB(x)y^2$, so that

$$(**) \qquad yxyx^2\phi(x) = xx^2\phi(x)y^2.$$

From (*) (with $y = x$), $x^2\phi(x) = \phi(x)x^2$ for all x in G.
Next invert the left-hand side, right-hand side, of (*) and multiply by (**).

$$\phi(x)^{-1}(yxy)^{-1}(yxy)x^2\phi(x) = y^{-2}\phi(x)^{-1}x^{-1}xx^2\phi(x)y^2.$$

But $\phi(x)$ commutes with x^2, so that

$$x^2 = y^{-2}x^2y^2.$$

Thus squares commute. Now set $x = t^2$ in (*): $yt^2y\phi(t)^2 = t^2\phi(t)^2y^2$, so that

$$yt^2y = t^2y^2, \qquad yt^2 = t^2y.$$

Thus the square t^2 of every element t is in the center of G. (*) can now be simplified to $xy = yx$, so that G is abelian.

F. W. Barnes rediscovered Levi's Theorem. G. P. Wene mentioned H. E. Bell, *The identity* $(xy)^n = x^n y^n$: *does it buy commutativity*?, Math. Mag., 55 (1982) 165–169.

Also solved by A. Bager (Denmark), F. W. Barnes (Kenya), M. Bencze (Romania), S. D. Bronn, P. L. Chabot, C.-N. Lee (student), L. Kuipers (Switzerland), J. J. Martinez, D. McCevitt (student), E. T. Ordman, G. P. Wene, D. Wiedemann, and E. T. Wong.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by January 31, 1985. The solver's full post-office address should be on each sheet.

6463. *Proposed by José M. Bayod, Santander, Spain.*

Let $S \subset R^n$ be a Lebesgue-measurable set with finite measure $\mu(S)$. Assume $f\colon S \to R$ is a real function that can be decomposed in the following way: $f = h \cdot g$, with g real, measurable and bounded (call $a = \inf g(S)$ and $b = \sup g(S)$), and h absolutely continuous on $[a,b]$. Then prove that f is integrable over S, and

$$\int_S f = \mu(S)h(b) - \int_a^b h'(t)\mu\left(g^{-1}[a,t]\right)dt.$$

6464. *Proposed by Roger Cooke, University of Vermont.*

What is the least upper bound of real numbers b such that there exists a continuous real-valued function $f(x)$ satisfying

$$xf(x) - \int_1^x \left(f(t) + t(f(t))^2\right)dt - b(\ln(x)) \to \infty \quad \text{as } x \to \infty?$$

6465. *Proposed by F. S. Cater, Portland State University.*

Prove that for each integer $n \geqslant 3$, there is a largest number h_n such that there exists a polynomial p of degree at most n which increases on the interval $(-h_n, h_n)$ and satisfies