Solution by D. Ž. Djoković, University of Belgrade, Yugoslavia. The existence is clear since we can take

$$K(f) = I(f) \equiv \int_0^1 f(x)dx \in \text{Range } (f).$$

In order to prove uniqueness of K we start from the identity

$$S^n f(x) = \frac{1}{2^n}\left[\sum_{k=0}^{2^{n-1}-1} f\left(\frac{k}{2^{n-1}} + \frac{x}{2^n}\right) + \sum_{k=1}^{2^{n-1}} f\left(\frac{k}{2^{n-1}} - \frac{x}{2^n}\right)\right],$$

which can be proved by induction. It follows that $S^n f(x) \to I(f)$ as $n \to \infty$ uniformly for $x \in [0, 1]$. That is, if $\epsilon > 0$

$$I(f) - \epsilon < S^n f(x) < I(f) + \epsilon, \qquad n > n_0(\epsilon), \qquad x \in [0, 1].$$

Consequently $K(f) = K(S^n f) \in (I(f) - \epsilon,\ I(f) + \epsilon)$, implying that $K(f) \equiv I(f)$.

Also solved by Robert Cohen, Roy O. Davies (England), Michael Edelstein, P. G. Engstrom, A. G. Heinicke, Sim Lasher, M. D. Mavinkurve (India), A. G. P. M. Nijet (Netherlands), and Stanton Philipp.

Notes. (1) Davies observes that $S^n f$ converges uniformly for every Riemann-integrable f and may converge uniformly for non-Riemann-integrable functions—e.g., the characteristic function of the set of all rational multiples of $\sqrt{2}$, wherein $S^n f \to 0$. For f integrable-R, the functional $I(f)$ is, clearly, not necessarily in the range of f.

(2) In an obvious fashion, the same result may be obtained using

$$Sf = \lambda f(\lambda x) + (1 - \lambda) f(1 - (1 - \lambda)x), \ 0 < \lambda < 1.$$

Subnets of Convergent Nets

5222 [1964, 802]. *Proposed by M. Rajagopalan and A. Wilansky, Lehigh University*

Must every convergent net in a metric space have a subnet whose range has at most one limit point?

Solution by E. T. Ordman, Princeton University. No. The following construction gives a counterexample in the Euclidean plane.

Let D be the class of finite subsets $s = \{a_1, a_2, \cdots, a_n\}$ of the interval $(0, 1)$. We may partially order D by letting $s \geqq t$ whenever $s \supset t$; any two elements are followed by their union, so $\geqq$ directs D. Any element of D follows at most finitely many others, and D is uncountable (since $\{a\} \in D$ for $0 < a < 1$), so any cofinal subset of D must be uncountable. We shall define a net T with domain D into the Euclidean plane which is one-to-one and which converges to zero. Then any subnet of T will include in its range images of the points of some cofinal subset of D; that is, the range of the subnet will be uncountable, and thus have an uncountable number of limit (in fact, condensation) points other than zero.

T may be constructed as follows. Let D_n be the subset of D consisting of n-element subsets of $(0, 1)$; D_n may be mapped one-to-one into $(0, 1)$ by writing each $s = \{a_1, a_2, \cdots, a_n\}$ in ascending order, writing each a_k as a decimal expansion (terminating if possible) $a_k = \cdot a_{1,k}a_{2,k}, \cdots$, and then letting $\delta(s) = \cdot a_{1,1}a_{1,2} \cdots a_{1,n}a_{2,1}a_{2,2} \cdots$. δ is then one-to-one into $(0, 1)$, in general not onto. We now introduce polar coordinates into the plane; (r, a) is the point a radians from the axis on the circle of radius r. Define $T(s) = (1/n, \delta(s))$ for $s \in D_n$; T is then one-to-one and eventually inside each circle of radius $1/n$, so T converges to zero.

Also solved by Ethan Akin, M. M. Chawla (India), Michael Edelstein, Hewitt Kenyon, J. R. Porter, and the proposers.

Editorial Note. As indicated by Chawla, the problem may also be resolved by the use of Theorem 6, p. 71 in J. L. Kelley, *General Topology.*

As posed originally, the problem asked for exactly one limit point, which admits the trivial example $\{(-1)^n\}$ whose range is $\{-1, 1\}$ and has no limit points. Happily the problem was accepted by the solvers as intended by the proposers.

The Torsion Subgroup of an Infinite Abelian Group

5223 [1964, 802]. *Proposed by C. R. MacCluer, University of Michigan*

Let L be the additive Abelian group of all ∞-tuples $(a_1, a_2, \cdots)$ where the nth entry is drawn from the integers modulo p^n, p a fixed prime, and let addition in L be coordinate-wise. Let G be the torsion subgroup and H the subgroup of all elements that have almost all zero entries. Show that H is not a direct summand of G. (This provides an example showing that purity of H does not imply that H is a direct summand even in the torsion case.)

Solution by Victor Keiser, University of Colorado. Let $\bar{g} = (a_1, a_2, \cdots) \in G$. Since $\bar{g}$ has finite order, almost all entries satisfy $(a_i, p) = p$. For all such a_i the equation $px = a_i$ has a solution, say h_i. Let $\bar{h} = (h_1, h_2, \cdots)$ where h_i is the solution of $px_i = a_i$ if it exists and $h_i = 0$ otherwise. Then $(\bar{g} - p\bar{h}) \in H$ because almost all its entries are zero. Thus $p(H + \bar{h}) = H + \bar{g}$, so we see that G/H is divisible.

Now suppose that H is a direct summand of G, say $G = H \oplus S$. Then $S \cong G/H$, so S must be divisible. But every element of $p^i S$ has a zero in the ith position, so $p^i S \neq S$ for some i. Hence S is not divisible.

Also solved by D. Ž. Djoković (Yugoslavia), Jack R. Porter, Burnett R. Toskey, and the proposer.

Representation of a Legendre Sum

5224 [1964, 802]. *Proposed by L. Carlitz, Duke University*

Let $\psi(a) = (a/p)$, the Legendre symbol. Show that if $abcd \not\equiv 0 \pmod{p}$,

$$S = \sum_{x,y,z=0}^{p-1} \psi(ax^2 + by^2 + cz^2 - 2dxyz) = -p\{\psi(a) + \psi(b) + \psi(c) + \psi(-abc)\}.$$